PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §4

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Lemma 1. *Let F be a complete ordered field. Let $A \subset F$ and $x \in F$.*

(a) *If A is bounded above and $x < \sup A$, then $x < a$ for some $a \in A$.*
(b) *If A is bounded below and $x > \inf A$, then $x > a$ for some $a \in A$.*

Proof. We prove **(a)**; the proof of **(b)** is analogous.

Suppose that $x \geq \sup A$. Then x is an upper bound for A. By definition of supremum, $\sup A \leq x$. This is the contrapositive of what we wished to prove. □

Exercise 1 (4.5)**.** Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Show that if $\sup S$ belongs to S, then $\sup S = \max S$.

Proof. Let $\alpha = \sup S$. Then $\alpha \geq s$ for all $s \in S$. Since $\alpha \in S$, we have $\alpha = \max S$. □

Exercise 2 (4.6)**.** Let S be a nonempty bounded subsets of $\mathbb{R}$. Show that $\inf S \leq \sup S$. What can be said if $\inf S = \sup S$?

Proof. Since S is nonempty, there exists $s \in S$. Then $\inf S \leq s$ and $s \leq \sup S$. By transitivity of order, $\inf S \leq \sup S$.

If $\inf S = \sup S$, then S contains only one element. □

Exercise 3 (4.7.a)**.** Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show if $S \subset T$, the $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Proof. Let $s \in S$. Then $s \in T$, so $\inf T \leq s$. Thus $\inf T$ is a lower bound for S, so $\inf T \leq \inf S$. Similarly, $\sup S \leq \sup T$. That $\inf S \leq \sup S$ is true is exercise 6. □

Exercise 4 (4.7.b)**.** Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Proof. Either $\max\{\sup S, \sup T\} = \sup S$ or $\max\{\sup S, \sup T\} = \sup T$.

Suppose that $\max\{\sup S, \sup T\} = \sup S$; in this case, $\sup T \leq \sup S$. Since $S \subset S \cup T$, we have $\sup S \leq \sup(S \cup T)$ by part (a).

Now let $x \in S \cup T$. Then x is either in S or T. If $x \in S$, then $x \leq \sup S$. If $x \in T$, then $x \leq \sup T \leq \sup S$. Thus $\sup S$ is an upper bound for $S \cup T$. Therefore $\sup(S \cup T) \leq \sup S$.

Since $\sup S \leq \sup(S \cup T)$ and $\sup(S \cup T) \leq \sup S$, it follows that $\sup S = \sup(S \cup T)$.

Finally, if $\max\{\sup S, \sup T\} = \sup T$, the above proof is valid, with the roles of S and T reversed. □

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Exercise 5 (4.8(b))**.** Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \leq t$ for every $s \in S$ and $t \in T$. Show that $\sup S \leq \inf T$.

Proof. Note that since S and T are nonempty, S is bounded above by an existing element of T and T is bounded below by an existing element of S. Thus $\sup S$ and $\inf T$ exist.

Suppose the conclusion is false; then $\inf T < \sup S$. By Lemma 1a, there exists $s \in S$ such that $\inf T < s$, By Lemma 1b, there exists $t \in T$ such that $t < s$. This is contrary to the assumption on S and T. □

Exercise 6 (4.10)**.** Show that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Proof. Let $b = \max\{a, \frac{1}{a}\}$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > b$. Since $a \leq b$, we have $a < n$. Also since $\frac{1}{a} \leq b$, we have $\frac{1}{a} < n$. Thus by Theorem 3.2.(vii), we have $\frac{1}{n} < a$. □

Exercise 7 (4.11)**.** Let $a, b \in \mathbb{R}$ such that $a < b$. Show that there exist infinitely many rational numbers between a and b.

Proof. Suppose not. The the set $S = (a, b) \cap \mathbb{Q}$ is finite, so it has a minimum, say $c = \min S$. But then Theorem 4.7 tells us that there exists $d \in \mathbb{Q}$ such that $a < d < c$. But then $d < b$, so $d \in S$. This contradicts that $c = \min S$. □

Define $\mathbb{I} = \mathbb{R} \smallsetminus \mathbb{Q}$. For the purposes of the next exercise, assume that $\mathbb{I}$ is nonempty. We will show this in the appendix.

Exercise 8 (4.12)**.** Let $a, b \in \mathbb{R}$. Show that if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$.

Proof. Since $\mathbb{I}$ is nonempty, let $\alpha \in \mathbb{I}$.

Let $q \in \mathbb{Q}$. Then $q \in \mathbb{R}$, and since $\alpha \in \mathbb{R}$ and $\mathbb{R}$ is a field, $q + \alpha \in \mathbb{R}$. Suppose that $q + \alpha \in \mathbb{Q}$; say $q + \alpha = p \in \mathbb{Q}$. Then $\alpha = p - q$, and since p and q are both in $\mathbb{Q}$, then so is $p - q$, because $\mathbb{Q}$ is a field. This contradicts the assumption on α. Thus $q + \alpha$ is irrational.

By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $q \in \mathbb{Q}$ such that $a - \alpha < q < b - \alpha$. Then $a < q + \alpha < b$, and $q + \alpha$ is irrational. Therefore, there exists an irrational number between any two real numbers. □

Exercise 9 (4.14)**.** Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let

$$S = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup S = \sup A + \sup B$.
(b) Show that $\inf S = \inf A + \inf B$.

Proof. We prove **(a)**; the proof for **(b)** is symmetric. It suffices to show that $\sup S \leq \sup A + \sup B$ and that $\sup A + \sup B \leq \sup S$.

Let $s \in S$. Then $s = a + b$ for some $a \in A$ and $b \in B$. Then $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound for S, so $\sup S \leq \sup A + \sup B$.

Suppose that $\sup S < \sup A + \sup B$. Then $\sup S - \sup B < \sup A$, so there exists $a \in A$ such that $\sup S - \sup B < a$. From this, $\sup S - a < \sup B$, so there exists $b \in B$ such that $\sup S - a < b$. Let $s = a + b \in S$. We have $\sup S < s$, a contradiction. Therefore $\sup A + \sup B \leq \sup S$. □

Exercise 10 (4.15)**.** Let $a, b \in \mathbb{R}$. Show that if $a \le b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \le b$.

Proof. We prove the contrapositive.

Suppose that $a > b$. By exercise 4.10, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a - b$. Thus $b + \frac{1}{n} < a$. □

Exercise 11 (4.16)**.** Show that $\sup\{r \in \mathbb{Q} \mid r < a\} = a$ for each $a \in \mathbb{R}$.

Proof. Let $a \in \mathbb{R}$, $A = \{r \in \mathbb{Q} \mid r < a\}$, and $s = \sup A$. We wish to show that $a = s$.

Suppose that $a < s$. Then there exists $r \in A$ such that $a < r < s$. This contradicts the definition of A.

On the other hand, suppose that $s < a$. By the density of $\mathbb{Q}$, there exists $r \in \mathbb{Q}$ such that $s < r < a$. Then $r \in A$. This contradicts the definition of s.

The only remaining possibility is that $a = s$. □

We now use the completeness axiom to prove that for every nonnegative real number a there exists a unique nonnegative real number b such that $b^2 = a$, which we denote by $\sqrt{a}$ (see related Exercise 6.6). By the Rational Roots Theorem, we know that there is no rational number whose square is equal to 2; thus this shows that $\mathbb{Q}$ is not complete, and that irrational numbers exist.

The plan of the proof is as follows. We wish to find a set of rational numbers such that its supremum is the square root of a. The natural set to consider is

$$S = \{x \in \mathbb{Q} \mid x^2 < a\}.$$

We are using $\mathbb{Q}$ here primarily for aesthetic reasons: we wish to construct an irrational number from the rational ones using the Completeness Axiom.

Let $b = \sup S$. We wish to show that $b^2 = a$. Thus we try to show that $b^2 \le a$ and $a \le b^2$. Each of these inequalities presents its own difficulties.

To show that $b^2 \le a$, we note that we can select an $s \in S$ as close to b as we like; thus their squares will be as close to b^2 as we like. If $a < b^2$, then one of these squares will be bigger than a, a contradiction.

To show that $a \le b^2$, assume that $b^2 < a$ and find a rational whose square is between b^2 and a. To do this, we first show that the set of square integers between 0 and 1 is dense in $[0, 1]$ by breaking up the interval into pieces whose endpoints are square rationals with denominators n^2 for any $n \in \mathbb{N}$. If n is large enough, the distance between any two of these endpoints is less than $\beta - \alpha$.

The next two propositions help with the inequality $b^2 \le a$.

Proposition 1. *Let $S \subset \mathbb{R}$ be a set of real numbers which is bounded above, and let $b = \sup S$. Then for every $n \in \mathbb{N}$ there exists $s \in S$ such that $b - s < \frac{1}{n}$.*

Proof. Otherwise, $b - \frac{1}{n}$ is an upper bound for S. □

Proposition 2. *Let $x, y \in \mathbb{R}$ such that $0 \le x$. Suppose that for every $n \in \mathbb{N}$, we have $0 \le x \le \frac{y}{n}$. Then $x = 0$.*

Proof. We prove the contrapositive.

Suppose that $x > 0$. We wish to show that there exists $n \in \mathbb{N}$ such that $\frac{y}{n} < x$.

Now either $y \le 0$ or $y > 0$. If $y \le 0$, then $\frac{y}{n} \le 0 < x$ for any $n \in \mathbb{N}$. If $y > 0$, then $0 < \frac{y}{x}$, so there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \frac{x}{y}$. Thus $\frac{y}{n} < x$. □

The next three propositions will give us the inequality $a \leq b^2$.

Proposition 3. *Let $q \in \mathbb{Q}$ be a positive rational number. Then there exists $n \in \mathbb{N}$ such that $1 - (\frac{n-1}{n})^2 < q$.*

Proof. Since $q \in \mathbb{Q}$, there exist $l, m \in \mathbb{Z}$ such that $q = \frac{l}{m}$, and since $q > 0$, we may choose $l, m > 0$. Thus $\frac{1}{m} \leq q$.

Let $n = 2m$. Then

$$\frac{2}{n} - \frac{1}{n^2} = \frac{1}{m} - \frac{1}{4m} < \frac{1}{m} \leq q;$$

so $-q < -\frac{2}{n} + \frac{1}{n^2}$. Adding 1 to both sides gives

$$1 - q < 1 - \frac{2}{n} + \frac{1}{n^2} = \frac{n^2 - 2n + 1}{n^2} = (\frac{n-1}{n})^2.$$

Therefore $1 - (\frac{n-1}{n})^2 < q$. □

Proposition 4. *Let $n, i \in \mathbb{N}$ with $0 < i < n$. Then*

$$(\frac{i}{n})^2 - (\frac{i-1}{n})^2 < 1 - (\frac{n-1}{n})^2.$$

Proof. Since $i < n$, we have $2i - 1 < 2n - 1$. Then

$$i^2 - (i-1)^2 = 2i - 1 < 2n - 1 = n^2 - (n-1)^2.$$

The result follows upon dividing by n^2. □

Proposition 5. *Let $\alpha, \beta \in \mathbb{Q}$ with $0 < \alpha < \beta$. Then there exists $\gamma \in \mathbb{Q}$ such that $\alpha < \gamma^2 < \beta$.*

Proof. First assume that $0 < \alpha < \beta < 1$.

Let $q = \beta - \alpha$; note that $q > 0$. By Proposition 3, there exists $n \in \mathbb{N}$ such that $1 - (\frac{n-1}{n})^2 < q$. Let i be the smallest integer such that $\beta < (\frac{i}{n})^2$; since $\beta < 1$, such an integer exists, and and $i \leq n$. Then $(\frac{i-1}{n})^2 < \beta$. Now by Proposition 4,

$$\beta - \alpha > 1 - (\frac{n-1}{n})^2 > (\frac{i}{n})^2 - (\frac{i-1}{n})^2 > \beta - (\frac{i-1}{n})^2;$$

subtracting β from both sides and multiplying by -1 gives

$$\alpha < (\frac{i-1}{n})^2.$$

Letting $\gamma = \frac{i-1}{n}$, we have $\alpha < \gamma^2 < \beta$.

Now drop the assumption that $\beta < 1$. Then there exists a natural number n such that $\beta < n^2$. Then $0 < \frac{\alpha}{n^2} < \frac{\beta}{n^2} < 1$, so there exists $\gamma \in \mathbb{Q}$ such that $\frac{\alpha}{n^2} < \gamma^2 < \frac{\beta}{n^2}$. Therefore $\alpha < n^2\gamma^2 < \beta$. □

Lemma 2. *Let $a, b \in \mathbb{R}$ and suppose that $a^2 \leq b^2$. Show that $|a| \leq |b|$.*

Solution #1. Since $a^2 \leq b^2$, we have $b^2 - a^2 \geq 0$ (O4). But for any $x \in \mathbb{R}$, we know that x^2 is positive (Thm 3.2.(iv)), so $x^2 = |x^2| = |x|^2$ (Thm 3.5.(ii)). Therefore $|b|^2 - |a|^2 \geq 0$. Factoring gives $(|b| - |a|)(|b| + |a|) \geq 0$. Since $|b| + |a| \geq 0$ (we can add inequalities by a combination of O3 and O4), we can divide by this term to get $|b| - |a| \geq 0$ (O5). Thus $|a| \leq |b|$. □

Solution #2. Assume that $|a| > |b|$. Multiply this by $|a|$ to get $|a|^2 > |a||b|$ (O5); multiply it by $|b|$ to get $|a||b| > |b|^2$. By transitivity (O3), $|a|^2 > |b|^2$. This proves the contrapositive of what we wished to show. □

Theorem 1. *Let $a \in \mathbb{R}$ with $a \geq 0$, and let*

$$S = \{x \in \mathbb{Q} \mid x^2 < a\}.$$

Then S is bounded above, and $(\sup S)^2 = a$.

Proof. Let $s \in S$. If $|s| \leq 1$, then $s < 1 + a$. If $|s| > 1$, then $s < s^2 < a < 1 + a$. In either case, $1 + a$ is greater than s, so $1 + a$ is an upper bound for the set S. Thus $\sup S$ exists; let $b = \sup S$. Since $0 \in S$, we know that $b \geq 0$. We show that $a \leq b^2$ and that $b^2 \leq a$.

Suppose that $b^2 < a$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that $b^2 < q < a$. By the Proposition 5, there exists $s \in \mathbb{Q}$ such that $b^2 < s^2 < a$. By definition of S, $s \in S$. But by the lemma, $b^2 < s^2 \Rightarrow b < s$, so b is not an upper bound for S. This contradiction shows that $a \leq b^2$.

Since $b = \sup S$, Proposition 1 tells us that for every $n \in \mathbb{N}$ there exists $s \in S$ such that $b - s < \frac{1}{n}$. Then $(b - s)(b + s) < \frac{b+s}{n}$. So

$$0 \leq b^2 - a < b^2 - s^2 < \frac{b + s}{n} < \frac{2b}{n}.$$

By Proposition 2, we have $b^2 - a = 0$, so $b^2 = a$. □

Corollary 1. *Let $a \in \mathbb{R}$ be a nonnegative real number. Then there exists a unique nonnegative real number $b \in \mathbb{R}$ such that $b^2 = a$.*

Proof. By Theorem 1, the polynomial equation $f(x) = x^2 - a$ has a root, say c. Then $-c$ is also a root, since $(-c)^2 = c^2 = a$. By a corollary to the division algorithm for polynomials, there are at most two roots. We see that if we let $b = |c|$, it is a unique positive root of $f(x)$. □

This justifies the notation $\sqrt{a}$. We have seen that there is no rational number whose square is 2; this shows that $\sqrt{2} \in \mathbb{I}$, and in particular, $\mathbb{I}$ is nonempty.

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