PRINCIPLES OF ANALYSIS
SOLUTIONS TO ROSS §9

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Exercise 1 (9.3)**.** Let (a_n) and (b_n) be sequences of real numbers and suppose that $\lim a_n = a$ and $\lim b_n = b$. Let

$$s_n = \frac{a_n^3 + 4a_n}{b_n^2 + 1}.$$

Show that

$$\lim s_n = \frac{a^3 + 4a}{b^2 + 1}.$$

Proof. In order to use Theorems 9.2 through 9.6, we have to verify that the components limits exist.

Since (a_n) converges to a, then (a_n^2) converges to a^2 by Theorem 9.4. Thus (a_n^3) converges to a^3 by another application of Theorem 9.4. Also $(4a_n)$ converges to $4a$ by Theorem 9.2. Thus $(a_n^3 + 4a_n)$ converges to $a^3 + 4a$ by Theorem 9.3.

Since (b_n) converges to b, then (b_n^2) converges to b^2 by Theorem 9.4. Since $\lim 1 = 1$ (proof left to reader), we have the $(b_n^2 + 1)$ converges to $b^2 + 1$ by Theorem 9.3.

Thus (s_n) converges to $\frac{a^3+4a}{b^2+1}$ by Theorem 9.6. □

We use the following lemma in Exercises 9.4 and 9.6.

Lemma 1. *Let* (s_n) *be a convergent sequence of real numbers. Then* $\lim s_n = \lim s_{n+1}$.

Proof. Let $L = \lim s_n$. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $|s_n - L| < \epsilon$ for all $n > N$. Now if $n > N$, then so is $n + 1$; thus $|s_{n+1} - L| < \epsilon$ for all $n > N$. □

Date: October 28, 2005.

Exercise 2 (9.4)**.** Let $s_1 = 1$ and for $n \geq 1$, let $s_{n+1} = \sqrt{s_n + 1}$. This defines a sequence $(s_n)_{n\in\mathbb{N}}$. Show that (s_n) converges, and that

$$\lim s_n = \frac{1+\sqrt{5}}{2}.$$

Point of Interest. Let $a, b \in \mathbb{R}$ with $a < b$. A *golden section* of $[a, b]$ is a point $c \in [a, b]$ with $c - a \geq b - c$ such that $\frac{b-a}{c-a} = \frac{c-a}{b-c}$. This common ratio is known as the *golden number*, and is denoted by φ.

Let $x = b - a$, $y = c - a$, and $z = b - c$; we have $x = y + z$ and $\varphi = \frac{x}{y} = \frac{y}{z}$. Thus $y^2 - zy - z^2$; by the quadratic formula, $y = \frac{z\pm\sqrt{z^2+4z^2}}{2} = z\frac{1\pm\sqrt{5}}{2}$. The negative solution is spurious, and taking the ratio $\frac{y}{z}$ cancels the z; thus

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

This number reappears in the context of regular pentagons, logarithmic spirals, and the Fibonacci sequence.

Note that φ is the positive solution to $x^2 - x - 1 = 0$. Thus $\varphi^2 = \varphi + 1$ and that $\frac{1}{\varphi} = \varphi - 1$. □

Solution to Exercise. To show that (s_n) converges, we use Ross Theorem 10.2, which states at bounded monotone sequences converge. Thus we show that (s_n) is increasing and bounded above by φ, that is, we show that $0 < s_n < s_{n+1} < \varphi$.

Proceed by induction on n. For $n = 1$, we have $s_1 = 1$ and $s_2 = \sqrt{2}$. Since $0 < 1 < \sqrt{2} < \varphi$, the base case holds.

By induction, assume that $0 < s_{n-1} < s_n < \varphi$. Note that $s_n = \sqrt{s_{n-1} + 1}$, so $s_n^2 = s_{n-1} + 1$, and $s_{n-1} = s_n^2 - 1$. Thus $0 < s_n^2 - 1 < s_n < \varphi$. Similarly, $s_n = s_{n+1}^2 - 1$, so $0 < s_n^2 - 1 < s_{n+1}^2 - 1 < \varphi$. Thus $0 < s_n < s_{n+1} < \sqrt{\varphi - 1} = \varphi$.

Therefore (s_n) is a bounded monotone sequence, and as such it converges. Let $s = \lim s_n$.

To show that $s = \varphi$, we use Ross Example 8.5, which states that if (a_n) is a convergent sequence of positive numbers, then $\lim \sqrt{a_n} = \sqrt{\lim a_n}$.

Let $t_n = s_{n+1}$. Then $\lim t_n = s$. To see this, let $\epsilon > 0$, and let N be so large that $|s_n - s| < \epsilon$ for all $n > N$. Then if $n > N$, we have $|t_n - s| = |s_{n+1} - s| < \epsilon$, since $n + 1 > N$. Thus (t_n) converges to s.

Note that $t_n = \sqrt{s_n + 1}$. Then

$$s = \lim t_n = \lim \sqrt{s_n + 1} = \sqrt{\lim s_n + 1} = \sqrt{s + 1}.$$

Thus $s^2 = s + 1$, so $s^2 - s - 1 = 0$. By the quadratic formula, $s = \frac{1\pm\sqrt{5}}{2}$. But since $s_n > 0$ for all $n \in \mathbb{N}$ and $1 - \sqrt{5} < 0$, we must have $s = \frac{1+\sqrt{5}}{2}$. □

Exercise 3 (9.6)**.** Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$. Show that if (x_n) converges, then $\lim x_n = 0$ or $\lim x_{n+1} = \frac{1}{3}$. In fact, show that if (x_n) converges, then there is life on Venus.

Proof. Suppose (x_n) converges and that $\lim x_n = a$. Then $\lim x_{n+1} = a$; thus $a = 3a^2$, so $a(3a-1) = 0$, whence $a = 0$ or $a = \frac{1}{3}$.

However, (x_n) does not converge. To see this, note that $x_n \geq n$ for all $n \in \mathbb{N}$. This follows by induction on n: for $n = 1$, it is immediate. Suppose that $n \geq 2$ and that it is true for $n-1$; that is, suppose that $x_{n-1} \geq n-1$. Then $x_n = 3x_{n-1}^2 \geq 3(n-1)^2 = 3n^2 - 6n + 3 > n$ (check this last inequality). Since n diverges to ∞, so does x_n by Exercise 9.9.(a).

Now assume that there is not life on Venus. Then (x_n) does not converge. Thus the contrapositive of the last statement is true. □

Exercise 4 (9.9.(a))**.** Let (s_n) and (t_n) be sequences in $\mathbb{R}$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $t_n \geq s_n$ for all $n \in \mathbb{N}, n > N_0$. Show that if $s_n = +\infty$, then $t_n = +\infty$.

Proof. To show that a sequence (t_n) diverges to $+\infty$, select an arbitrary (think "large") real number, and find $N \in \mathbb{N}$ such that $t_n > M$ for all $n > M$.

Let $M > 0$ and let N_1 be so large that $s_n > M$ for all $n > N_1$. Let $N = \max\{N_0, N_1\}$. Then if $n > N$, $M < s_n < M$. Thus $t_n \to \infty$. □

Exercise 5 (9.10.(a))**.** Let (s_n) be a sequence in $\mathbb{R}$ and let $k \in \mathbb{R}$. Show that if $lims_n = +\infty$ and $k > 0$, then $\lim(ks_n) = +\infty$.

Proof. This is a particular case of Thm 9.9. Let $t_n = k$ for all $n \in \mathbb{N}$. Then $\lim t_n = k > 0$, so $\lim ks_n = \lim s_n t_n = +\infty$. □

Exercise 6 (9.11)**.** Let (s_n) and (t_n) be sequences in $\mathbb{R}$ such that $\lim s_n = +\infty$ and (t_n) satisfies one of the following:
(a) $\inf\{t_n \mid n \in \mathbb{N}\} > -\infty$;
(b) $\lim t_n > -\infty$;
(c) (t_n) is bounded.
Show that $\lim(s_n + t_n) = +\infty$.

Proof. Let $T = \{t_n \mid n \in \mathbb{N}\}$; clearly T is nonempty. The condition that $\inf T > -\infty$ is the same as saying that T is bounded below, so $\inf T$ exists as a real number.

Let $M \in \mathbb{R}$ and let $t = \inf t$. Since $s_n \to +\infty$, let N be so large that $s_n > M - t$ for all $n > N$. Then for $n > N$,

$$s_n + t_n \geq s_n + t > (M - t) + t = M.$$

Thus $s_n + t_n \to +\infty$.

Now if $\lim t_n > -\infty$, we see that (t_n) is bounded below. If (t_n) converges, then it is bounded. If $\lim t_n = +\infty$, then either 0 is a lower bound for T or $t_n > 0$ for all but finitely many n, and the infimum of T is the minimum of the set $\{t \in T \mid t < 0\}$.

Finally if (t_n) is bounded, then it is bounded below. □

Exercise 7 (9.12)**.** Let (s_n) be a sequence in $\mathbb{R}$ such that $s_n \neq 0$ for all $n \in \mathbb{N}$, and let $t_n = |\frac{s_{n+1}}{s_n}|$. Suppose that (t_n) converges to L.
(a) Show that if $L < 1$, then $\lim s_n = 0$.
(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.

Proof.
(a) Suppose that $L < 1$. Note that since $t_n > 0$ for all $n \in \mathbb{N}$, we have $L \geq 0$ by Exercise 8.9.(a). Since we wish to show that $s_n \to 0$, it suffices to assume that $s_n > 0$ for all $n \in \mathbb{N}$. This is because a sequence converges to zero if and only if its absolute value converges to zero by Exercise 8.6.(a). Saying this simply allows us to avoid writing lots of absolute value signs.

Let $\epsilon > 0$. Since $L < 1$, there exists $a \in \mathbb{R}$ such that $L < a < 1$. Since $t_n \to L$, there exists $N_0 \in \mathbb{N}$ such that $|\frac{s_{n+1}}{s_n} - L| < a - L$. Since $s_n > 0$ for all $n \in \mathbb{N}$, we have $\frac{s_{n+1}}{s_n} - L < a - L$, so $\frac{s_{n+1}}{s_n} < a$, and $s_{n+1} < as_n$ for all $n > N_0$.

Claim: $s_n < a^{n-N_0} s_{N_0}$ for $n > N_0$.
We prove this by induction, and note that we have the base case already. By induction, we assume that $n > N_0 + 1$ and that $s_{n-1} < a^{n-1-N_0} s_{N_0}$. Multiply both sides by a, which is positive, to get $as_{n-1} < a^{n-N_0} s_{N_0}$. Now $as_{n-1} > s_n$, so $s_n < a^{n-N_0} s_{N_0}$ by transitivity.

By Example 9.7.(b), we see that $a^n \to 0$ as $n \to \infty$. Let N_1 be so large that $|a^n - 0| < \frac{\epsilon}{s_{N_0}}$ for all $n > N_1$. Let $N = N_0 + N_1$. Then for $n > N$, we have

$$\begin{aligned} |s_n - 0| &= s_n \\ &< a^{n-N_0} s_{N_0} \\ &< \frac{\epsilon}{N_0} s_{N_0} \text{ because } n - N_0 > N_1 \\ &= \epsilon. \end{aligned}$$

Therefore $\lim_{n\to\infty} s_n = 0$.

(b) Suppose that $L > 1$. Then $\lim t_n^{-1} = L^{-1} < 1$ by Theorem 9.5. Now

$$t_n^{-1} = \frac{|s_n|}{|s_{n+1}|} = \frac{|s_{n+1}|^{-1}}{|s_n|^{-1}};$$

by part (a), we know that $|s_n|^{-1} \to 0$ as $n \to \infty$. Therefore $\lim |s_n| = +\infty$ by Theorem 9.10. □

Exercise 8 (9.15)**.** Show that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Proof. We apply Exercise 9.12.
Let $s_n = \frac{a^n}{n!}$, and let

$$t_n = \frac{s_{n+1}}{s_n} = \frac{a^{n+1}(n)!}{a^n(n+1)!} = \frac{a}{n}.$$

Then $\lim t_n = a \lim \frac{1}{n} = 0 < 1$, so by Exercise 9.12, we have $\lim s_n = 0$. □

Exercise 9 (9.16.(a))**.** Let $a_n = \frac{n^4+8n}{n^2+9}$. Show that $\lim a_n = +\infty$.

Proof. By Theorem 9.10, it suffices to show that $\lim a_n^{-1} = 0$, because (a_n) is a sequence of positive numbers.

Now $a_n^{-1} = \frac{n^2+9}{n^4+8n} = (\frac{1}{n^2} + \frac{9}{n^4})/(1 + \frac{8}{n^3})$. Using Theorem 9.3, we see that the numerator of this fraction converges to 0 and the denominator converges to 1. Since the denominator does not converge to 0, we can use Theorem 9.6 to conclude that $\lim a_n^{-1} = \frac{0}{1} = 0$. Thus $\lim a_n = +\infty$ by Theorem 9.10. □

Exercise 10 (9.16.(b))**.** Let $b_n = \frac{2^n}{n^2} + (-1)^n$. Show that $\lim b_n = +\infty$.

Proof. First we let $s_n = \frac{n^2}{2^n}$ and show that $\lim s_n = 0$. Consider

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)^2 2^n}{n^2 2^{n+1}} = \frac{1}{2} + \frac{1}{n} + \frac{1}{2n^2}.$$

By Theorem 9.3, this converges to $\frac{1}{2} < 1$. Then by Exercise 9.12.(a), (s_n) converges to 0. Since this is a sequence of positive numbers, we have that $\lim s_n^{-1} = \lim \frac{2^n}{n^2} = +\infty$.

Since the sequence $((-1)^n)$ is bounded, we can now apply Exercise 9.11 to see that

$$\lim b_n = \lim(s_n^{-1} + (-1)^n) = +\infty.$$

□

Exercise 11 (9.16.(c))**.** Let $c_n = \frac{3^n}{n^3} - \frac{3^n}{n!}$. Show that $\lim c_n = +\infty$.

Proof. Let $s_n = \frac{3^n}{n^3}$ and let $u_n = -\frac{3^n}{n!}$. By Exercise 9.11, it suffices to show that $s_n \to +\infty$ and (u_n) is bounded. Now $\lim u_n = 0$ by Exercise 9.15. By Theorem 9.10, it suffices to show that $\lim s_n^{-1} = 0$. This can be shown by the same method as in part (b). □

We can solve 9.1.(a), 9.1.(b), 9.1.(c), 9.16.(a), and many others, with the following proposition.

Proposition 1. *Let* $f(x) = a_s x^s + \cdots + a_1 x + a_0$ *and* $g(x) = b_t x^t + \cdots + b_1 x + b_0$ *be polynomial functions with real coefficients such that* $a_s \neq 0$ *and* $b_t > 0$. *Let* $r_n = \frac{f(n)}{g(n)}$ *if* $g(n) \neq 0$ *and* $r_n = c$ *if* $g(n) = 0$, *where* c *is any real number. Then*

$$\lim r_n = \begin{cases} 0 & \textit{if } s < t; \\ \frac{a_s}{b_t} & \textit{if } s = t; \\ \operatorname{sgn}(a_m)\infty & \textit{if } s > t. \end{cases}$$

Proof. First note that, for any positive integer m, we have $\lim_{n\to\infty} n^{-m} = 0$. To see this, let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{\epsilon} < N$; then $\frac{1}{N} < \epsilon$. For $n > N$, we have

$$|n^{-m} - 0| = \frac{1}{n^m} \leq \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Let n be any positive integer; we may multiply $\frac{f(n)}{g(n)}$ by $\frac{1/n^t}{1/n^t}$ to obtain

$$r_n = \frac{a_s n^{s-t} + a_{s-1} n^{s-t-1} \cdots + a_0 n^{-t}}{b_t + b_{t-1} n^{-1} \cdots + b_0 x^{-t}}.$$

Let $s_n = a_s n^{s-t} + \cdots + a_0 n^{-t}$ and let $t_n = b_t + \cdots + b_0 x^{-t}$; then $r_n = \frac{s_n}{t_n}$. Now

$$\lim t_n = b_t + b_{t-1} \lim n^{-1} + \cdots + b_0 \lim x^{-t} = b_t;$$

taking the limit inside of the sum is justified by the fact that the individual limits of the summands each exist. If $s \leq t$, then $s - t - i < 0$ for $i = 0, 1, \ldots, s$, so the limits of the summands of s_n exist and

$$\lim s_n = a_s \lim n^{s-t} + \cdots + a_0 \lim n^{-t} = 0.$$

If $s < t$, then $\lim s_n = 0$, and if $s = t$, we have $\lim s_n = a_s$.

Still assuming that $s \leq t$, since (s_n) and (t_n) are convergent in this case, we have

$$\lim r_n = \frac{\lim s_n}{\lim t_n} = \begin{cases} \frac{a_s}{b_t} \text{ if } s = t; \\ 0 \text{ if s ¡ t.} \end{cases}$$

Now suppose that $s > t$, and assume that $a_s > 0$. By the above, we know that the sequence $(\frac{1}{r_n}) = (\frac{g(n)}{f(n)})$ converges to 0. By Theorem 9.10, (r_n) diverges to $+\infty$.

Finally, if $a_s < 0$, then $\lim r_n = -\lim -r_n = -\infty$ by Exercise 9.10.(b). □

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