MATH 3063 ABSTRACT ALGEBRA Project 1 Name:
PROF. PAUL BAILEY February 11, 2008

Due Friday, February 15, 2008.

Write directly on this worksheet, and turn in the completed worksheet. Use the definitions. Follow the frameworks. Fill in the blanks of the outlined proofs, or fill in the gaps of the partially outlined proofs. Your proof MUST USE THE GIVEN SENTENCES.

Definition 1. Let $f : A \to B$.

If $C \subset A$, the *image* of C under f is

$$f(C) = \{b \in B \mid b = f(c) \text{ for some } c \in C\}.$$

If $D \subset B$, the *preimage* of D under f is

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}.$$

We say that f is *injective* if, for every $a_1, a_2 \in A$, we have

$$f(a_1) = f(a_2) \quad \Rightarrow \quad a_1 = a_2.$$

We say that f is *surjective* if

$$\forall b \in B \; \exists a \in A \quad \ni \quad f(a) = b.$$

We say that f is *bijective* if it is injective and surjective.

Type 1. Let A and B be sets. Show that $A \subset B$.

Method. Let $a \in A$. [work; use the defining property of A] Thus $a \in B$. Therefore $A \subset B$. □

Type 2. Let A and B be sets. Show that $A = B$.

Method. We show that $A \subset B$ and $B \subset A$.
($A \subset B$) Let $a \in A$. [work] Thus $a \in B$.
($B \subset A$) Let $b \in B$. [work] Thus $b \in A$.
Since $A \subset B$ and $B \subset A$, we have $A = B$. □

Type 3. Let $f : A \to B$. Show that f is injective.

Method. Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. [work; using definition of f, show that $a_1 = a_2$] Therefore $a_1 = a_2$. Since $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, f is injective. □

Type 4. Let $f : A \to B$. Show that f is surjective.

Method. Let $b \in B$. [work; using definition of f, find a such that $f(a) = b$] Therefore $f(a) = b$. Since $\forall b \in B \exists a \in A \ni f(a) = b$, f is surjective. □

Type 5. Let p and q be propositions. Show that $p \Leftrightarrow q$.

Method. We show that $p \Rightarrow q$ and $q \Rightarrow p$.
($p \Rightarrow q$) [work]
($q \Rightarrow p$) [work]
Since $p \Rightarrow q$ and $q \Rightarrow p$, we have $p \Leftrightarrow q$. □

Problem 1. Let $f : A \to B$ and let $D_1, D_2 \subset B$. Show that $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

Proof. We show containment in both directions.

($\subset$) Let $x \in f^{-1}(D_1 \cap D_2)$.

Then ______________ $\in D_1 \cap D_2$.

Thus $f(x) \in$ ______________ and $f(x) \in$ ______________.

Thus $x \in$ ______________ and $x \in$ ______________.

Therefore, $x \in$ ______________.

($\supset$) Let $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$.

Then $x \in$ ______________ and $x \in$ ______________.

Thus ______________ $\in D_1$ and ______________ $\in D_2$.

Thus $f(x) \in$ ______________ $\cap$ ______________.

Therefore, $x \in$ ______________. □

Problem 2. Let $f : A \to B$ and let $D_1, D_2 \subset B$. Show that $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

Proof. We show containment in both directions.
($\subset$) Let $x \in f^{-1}(D_1 \cup D_2)$; we wish to show that $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$.

Therefore $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$.

($\supset$) Let $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$; we wish to show that $x \in f^{-1}(D_1 \cup D_2)$.

Therefore $x \in f^{-1}(D_1 \cup D_2)$. □

Problem 3. Let $f : A \to B$ and $g : B \to C$. Suppose that f is surjective and $g \circ f$ is injective. Show that g is injective.

Proof. Let $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$. We wish to show that $b_1 = b_2$.

Since f is surjective, there exist $a_1, a_2 \in$ ______________ such that

$f(a_1) =$ ______________ and $f(a_2) =$ ______________.

Applying g to these equations gives $g(f(a_1)) =$ ______________ and $g(f(a_2)) =$ ______________.

But $g(b_1) = g(b_2)$, and since $g \circ f$ is injective, $a_1 =$ ______________.

Thus $f(a_1) =$ ______________, that is, $b_1 = b_2$.

Therefore g is injective. □

Problem 4. Let $f : A \to B$ and $g : B \to C$. Suppose that g is injective and $g \circ f$ is surjective. Show that f is surjective.

Proof. Let $b \in B$. We wish to find $a \in A$ such that $f(a) = b$.

Let $c = g($______________$)$.

Since $g \circ f$ is surjective, there exists $a \in A$ such that ______________ $= c$,

that is, $g(f(a)) = g(b)$.

Since g is injective, ______________ $= b$.

Therefore f is surjective. □

Problem 5. Let X be a set and let $\mathcal{P}(X)$ denote the power set of X. Define a relation $\equiv$ on $\mathcal{P}(X)$ by

$$A \equiv B \quad \Leftrightarrow \quad \text{there exists a bijective function } f : A \to B.$$

Show that $\equiv$ is an equivalence relation.

Proof. We show that $\equiv$ is reflexive, symmetric, and transitive.
(*Reflexivity*) Let $A \subset X$. Consider the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) =$ ______________.

This function is obviously ______________. Therefore, $A \equiv A$.

(*Symmetry*) Let $A, B \subset X$ such that $A \equiv B$. Then there exists a ______________ function $f : A \to B$.

Since f is ______________, there exists an inverse function $f^{-1} : B \to A$ which is also ______________. Therefore, $B \equiv A$.

(*Transitivity*) Let $A, B, C \subset X$ such that $A \equiv B$ and $B \equiv C$.

Then there exist ______________ function $f : A \to B$ and $g : B \to C$.

Since the ______________ of a ______________ functions is ______________,

the function $g \circ f : A \to C$ is ______________. Therefore, $A \equiv C$. □