THE COSINE OF 72°

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Problem 1. Compute the cosine of 72°.

Solution via Trigonometry. Let $\theta = 18^\circ$; then $\sin\theta = \cos 72^\circ$. Now $5\theta = 90^\circ$, so $3\theta = 90^\circ - 2\theta$; therefore

$$\sin 3\theta = \cos 2\theta.$$

Use the sine sum of angles formula on the left:

$$\sin 2\theta \cos\theta + \sin\theta\cos 2\theta = \cos 2\theta.$$

Subtract $\sin\theta\cos 2\theta$ from both sides:

$$\sin 2\theta\cos\theta = \cos 2\theta(1 - \sin\theta).$$

Use the sine double angle formula:

$$2\sin\theta\cos^2\theta = \cos 2\theta(1 - \sin\theta).$$

Apply the cosine squared and cosine double angle identities:

$$2\sin\theta(1 - \sin^2\theta) = (1 - 2\sin^2\theta)(1 - \sin\theta).$$

Divide by $(1 - \sin\theta)$ to get

$$2\sin\theta(1 + \sin\theta) = 1 - 2\sin^2\theta.$$

Make a polynomial in $\sin\theta$:

$$4\sin^2\theta + 2\sin\theta - 1 = 0.$$

Apply the quadratic formula to find that

$$\begin{aligned}\sin\theta &= \frac{-2 \pm \sqrt{4 + 16}}{8} \\ &= \frac{-1 \pm \sqrt{5}}{4}.\end{aligned}$$

Since 18° is in the first quadrant, the sine must be positive; we conclude that

$$\boxed{\cos 72^\circ = \sin 18^\circ = \frac{-1 + \sqrt{5}}{4}.}$$

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Solution via Complex Numbers. First we note that $72^\circ = \frac{360^\circ}{5} = \frac{\pi}{5}$ radians. It is clear from the geometry, and can be shown analytically, that $\cos\frac{\pi}{5} = \sin\frac{4\pi}{5}$, so this is the number we seek.

We wish to use complex numbers on the unit circle to solve this problem, Let $\alpha = e^{2\pi i}5$; we wish to find $\cos\frac{\pi}{5} = \Re\alpha$.

The complex number α is a primitive 5^{th} root of unity, and so α satisfies the polynomial equation $x^5 - 1 = 0$. Clearly 1 is a root of $x^5 - 1$; factor this out to obtain $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$. Now α is a root of the latter factor, so $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$.

Note that α^4 is the complex conjugate of α and α^2 is the complex conjugate of α^3. Thus, $\Re\alpha = \frac{1}{2}(\alpha + \alpha^4)$ and $\Re\alpha^2 = \frac{1}{2}(\alpha^2 + \alpha^3)$. Let $\zeta_1 = (\alpha + \alpha^4)$ and $\zeta_2 = (\alpha^2 + \alpha^3)$; we have $\zeta_1 + \zeta_2 = -1$. Compute

$$\zeta_1\zeta_2 = \alpha^3 + \alpha^4 + \alpha^6 + \alpha^7 = \alpha^3 + \alpha^4 + \alpha + \alpha^2 = -1.$$

Define the quadratic function

$$f(x) = (x - \zeta_1)(x - \zeta_2) = x^2 - (\zeta_1 + \zeta_2) + \zeta_1\zeta_2 = x^2 + x - 1.$$

The roots of f are given by the quadratic formula to be

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Geometrically, it is clear that $\zeta_1 > 0$ and $\zeta_2 < 0$; thus

$$\zeta_1 = \frac{-1 + \sqrt{5}}{2} \quad \text{and} \quad \zeta_2 = \frac{-1 - \sqrt{5}}{2}.$$

Thus

$$\boxed{\cos 72^\circ = \Re\alpha = \frac{-1 + \sqrt{5}}{4}.}$$

□

Solution via the Golden Triangle. Consider an isosceles triangle $\triangle ABC$ with the property that the equal angles $\angle ABC$ and $\angle ACB$ are twice the other angle $\angle BAC$. Let $\alpha = \angle BAC$ and $\beta = \angle ABC = \angle ACB$, so that $\beta = 2\alpha$. Then $5\alpha = 180^\circ$, so $\alpha = 36^\circ$ and $\beta = 72^\circ$. We wish to find $\cos\beta$.

Bisect $\angle ABC$ and let D be the point on $\overline{AC}$ such that $\angle BCD = \alpha$. Now $\triangle DAB$ is isosceles, with $|DA| = |BA|$, and $\triangle BCD$ is similar to $\triangle ABC$. Let $x = |AB|$, $y = |BC|$, and $z = |CD|$, so that $x = y + z$. Moreover, $\frac{x}{y} = yz$ by similarity. Thus $y^2 = xz = (y+z)z = yz + z^2$, so $y^2 - yz - z^2 = 0$, and by the quadratic formula,

$$y = \frac{z + \sqrt{z^2 + 4z^2}}{2} = z\frac{1 + \sqrt{5}}{2}.$$

Thus

$$\cos\beta = \frac{z}{2y} = \frac{1}{1 + \sqrt{5}} = \frac{-1 + \sqrt{5}}{5 - 1}.$$

Conclude that

$$\boxed{\cos 72^\circ = \frac{-1 + \sqrt{5}}{4}.}$$

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