History of Mathematics (Math 4123)
Practice Mathematical Problems

In construction problems, describe each step, and draw all steps with a straight-edge and compass, labeling each point significant for the construction. Explain why your construction works.

1 Easier

Problem 1. Write 1070 in Mayan.

Problem 2. Find the base six radix expansion of $\frac{271}{54}$.

Problem 3. (Babylonian Fractions)
Let $x = \frac{5}{72}$ and $b = 60$. Find the base b radix expansion of x. (Hint: $x = \frac{5a}{72a}$, where $72a$ is a power of 60.)

Problem 4. Using a straight edge and compass, construct the circle passing through three given points. Label the original points, all necessary constructed points, and describe exactly how the constructed points were created.

Problem 5. (Greek Geometry)
Find the area of a regular octagon inscribed in a unit circle.

Problem 6. The key definition of Eudoxus' theory of proportion is laid out in Euclid's Elements.

Elements Book V Definition 5
Magnitudes are said to be in the same *ratio*, the first to the second and the third to the fourth, when, if any *equimultiples* whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples *alike exceed*, are *alike equal to*, or alike *fall short of*, the latter equimultiples respectively taken in corresponding order.

Let $a, b, c, d \in \mathbb{R}$ be positive real numbers. Consider the proposition:

$$\frac{a}{b} = \frac{c}{d} \quad \Leftrightarrow \quad \forall m, n \in \mathbb{N}, \begin{cases} ma > nb \Leftrightarrow mc > nd; \\ ma = nb \Leftrightarrow mc = nd; \\ ma < nb \Leftrightarrow mc < nd. \end{cases}$$

(a) Identify the italicized words in the definition with the exact mathematical symbols in the proposition.

(b) Prove the proposition from a modern perspective.

Problem 7. (Euclidean Algorithm)
Let $m = 80$ and $n = 167$. Find $x, y, d \in \mathbb{Z}$ such that $d = \gcd(m, n)$ and

$$mx + ny = d.$$

Problem 8. (Diophantus' Theorem)
Let $a, b, c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$. Show that if a is odd, then $b + c$ is a perfect square.

Problem 9. Find the area of a regular octagon inscribed in the unit circle.

Problem 10. Consider the cubic curve with equation

$$y^2 = x^3 - 3x + 1.$$

Find a rational point on the curve other than $(0, \pm 1)$.

Problem 11. Let $n \in \mathbb{Z}$ with $n \geq 2$. Let $a, b, c, d \in \mathbb{Z}$ with $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$.
Show that $ab \equiv cd \pmod{n}$.

Problem 12. Find $c \in \mathbb{Z}$ with $0 \le c < 221$ such that $c \equiv 7 \pmod{13}$ and $c \equiv 11 \pmod{17}$.

Problem 13. Let $m = 41$, $n = 61$, $a = 21$, and $b = 31$.

(a) Find x and y so that $mx + ny = 1$.

(b) Find $c \in \mathbb{Z}$ with $0 \le c < 3501$ such that $c \equiv a \pmod{m}$ and $c \equiv b \pmod{n}$.

Problem 14. Define a sequence for real number (G_n) by $G_1 = 1$, $G_2 = 1$, and $G_{n+2} = 3G_n + G_{n+1}$. Let (a_n) be the sequence defined by $a_n = \frac{G_{n+1}}{G_n}$.

(a) Compute the first 5 terms of (G_n).

(b) Compute the first 5 terms of (a_n).

(c) Write a_{n+1} in terms of a_n.

(c) Compute $\lim a_n$.

Problem 15. Consider the cubic equation

$$x^3 + 3x^2 + 6x + 7 = 0.$$

(a) Substitute $x = y - 1$ to obtain an equation without a y^2 term.

(b) Use the method of Tartaglia to compute y which satisfies this equation.

(c) Find x which satisfies the original equation.

Problem 16. (Titles)
Indicate the author of each manuscript. Choose from these authors:
Archimedes, Appolonius, Cavalieri, Descartes, Diophantus, Euclid, Fibonacci, Gauss, Napier, Newton.

(a) *Principia Mathematica*

(b) *Liber Abaci*

(c) *Arithmetica*

(d) *The Elements*

(e) *La geometrie*

(f) *Disquisitiones arithmeticae*

(g) *Conic Sections*

(h) *Geometrica indivisibilibus*

(i) *On the Sphere and the Cylinder*

Problem 17. (Archimedes)
To compute π, Archimedes found the areas of many regular polygons. Find the area of a regular dodecagon inscribed in the unit circle.

Problem 18. (Diophantus)
To find Pythagorean triples, Diophantus consider the intersection of the unit circle with a line through $(-1, 0)$ with rational slope. Find the Pythagorean triple (a, b, c), with $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and $a^2 + b^2 = c^2$, that this technique produces when the slope of the line is $\frac{3}{5}$.

Problem 19. (Tartaglia)
To solve equations of the form $x^3 + mx = n$, Tartaglia set $x = t - u$ and used the substitutions $m = 3tu$ and $n = t^3 - u^3$. Apply this technique to find the solutions to $x^3 + 9x = 20$.

Problem 20. (Descartes)
To compute tangents, Descartes used the discriminant of a quadratic equation to find the circle centered on the x-axis and tangent to a given curve at a given point. Use this technique to find the center $(a, 0)$ of such a circle, when the equation is $2x^2 - y^2 = 1$, the point is $(1, 1)$, and $a > \frac{1}{2}$.

Problem 21. (Leibnitz)
To compute the sum of the reciprocals of the triangular numbers, Leibnitz used a telescoping sum. Reproduce this argument.

Problem 22. (Euler)
To compute the sum of the reciprocals of the square numbers, Euler considered the zeros of the function $\sin x/x$ to produce the equation

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

He then equated the coefficients of the x^2 term on both sides and obtained $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(a) Compute the sum of the reciprocals of the even square numbers by substituting $x \mapsto x/2$.

(b) Compute the sum of the reciprocals of the odd square numbers by subtraction.

Problem 23. (Pythagorean Triples)
The Babylonians generated tables of Pythagorean triples (a, b, c) such that a is sexagesimally regular. Euclid's *Elements* supplied a technique for computing Pythagorean triples using the equations

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2.$$

Diophantus proved that this produces *all* Pythagorean triples.

Thus the following function generates Pythagorean triples:

$$\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N} \quad \text{by} \quad \phi(u, v) = (2uv, u^2 - v^2, u^2 + v^2).$$

Set

$$S = \{n \in \mathbb{N} \mid 1 \leq n \leq 10 \text{ and } n \text{ is decimally regular}\};$$
$$U = \{(u, v) \in S \times S \mid v < u \text{ and } \gcd(u, v) = 1\}.$$

(a) Find S.

(b) Find U.

(c) Find $\phi(U)$.

Problem 24. (Regular Solids)
The *regular solids* were studied by the Pythagoreans, the Platonists, and Euclid.

(a) List the regular solids. State the type of regular polygon from which each solid is constructed. Find the number of faces F, the number of edges E, and the number of vertices V. Compute $F - E + V$.

(b) Luca Pacioli (1509) used three intersecting golden rectangles to construct a regular solid whose faces are equilateral triangles with sides of length one. Use this construction to find the radius of a sphere in which such a solid can be transcribed.

Problem 25. (Diophantine Geometry)
A *rational curve* is the set of solutions to a polynomial equation in two variables whose coefficients are rational numbers. A *rational point* on a curve is a solution whose coordinates are rational numbers.

Diophantus (Alexandria, 2^{nd} century A.D.) realized that, given two rational points on a cubic curve, the slope between them would be rational, and so the third point of intersection between the line and the curve would produce another rational point.

Consider the curve given by the equation

$$y^2 = x^3 - 4x + 9.$$

By trying small values for x, find four rational points on this curve. Select two points such that the slope of the line between them is 3. Compute this line. Intersect this line with the curve to find two additional rational points.

Problem 26. (Congruence)
Euclid's *Elements* contains a description of the Euclidean algorithm for find x, y such that

$$mx + ny = \gcd(m, n).$$

The proof of the *Chinese Remainder Theorem* uses this fact to produce solutions to systems of congruences of the form

$$a \equiv c \pmod{m};$$
$$b \equiv c \pmod{n}.$$

Let $m = 17$, $n = 37$, $a = 7$, and $b = 11$.

(a) Find x and y such that $mx + ny = 1$.

(b) Find c with $0 \leq c < mn$ such that $a \equiv c \pmod{m}$ and $b \equiv c \pmod{n}$.

Problem 27. Constructibility
Let A be a set of points in a plane $\mathcal{P}$. Let $\mathcal{L}(A)$ be the set of all lines in $\mathcal{P}$ which pass through at least two points in A, and let $\mathcal{C}(A)$ be the set of all circles in $\mathcal{P}$ pass through a point in A and whose center is a different point in A. Let $\mathcal{O}(A) = \mathcal{L}(A) \cup \mathcal{C}(A)$. Define

$$S(A) = \{z \in \mathcal{P} \mid z \in O_1 \cap O_2 \text{ for some } O_1, O_2 \in \mathcal{O}(A)\}.$$

(a) If A contains one point, how large is $S(A)$?

(b) If A contains two points, how large is $S(A)$?

(c) If A contains three collinear equally spaced points, how large is $S(A)$?

(d) If A contains three collinear unequally spaced points, how large is $S(A)$?

(e) If A contains the vertices of an equilateral triangle, how large is $S(A)$?

(f) If A contains the vertices of an acute isosceles triangle, how large is $S(A)$?

(g) If A contains the vertices of an obtuse isosceles triangle, how large is $S(A)$?

Include a drawing to justify each case.

Problem 28. Given two points A, B in a plane, describe all steps necessary to construct a point C such that $AC \perp AB$ and $\triangle ABC$ is an isosceles triangle.

Problem 29. Let $d = \gcd(728, 231)$. Use that Euclidean Algorithm to find $d, x, y \in \mathbb{Z}$ such that

$$mx + ny = d.$$

2 Harder

Problem 30. Let m and n be integers with $m, n \geq 3$. Let $d = \gcd(m, n)$ and let $k = \frac{mn}{d}$. Given a regular m-gon and a regular n-gon, construct a regular k-gon.

Problem 31 (Regarding Archimedes)**.** Let P be a regular n-gon and let O be its center. Let A and B be consecutive vertices on P and and assume that $|OA| = 1$. Let M be the midpoint between A and B. Find $|OM|$ as a function of n.

Problem 32. Solve the following equations for the positive integers n and b.

(a) $n = (13425)_b = (4115)_{2b}$

(b) $n = (1234)_b = (532)_{2b-1}$

(See Eves Problem Study 1.8.)

Problem 33. A *Pythagorean triple* is and ordered triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$.

(a) Show that there exists a Pythagorean triple (a, b, c) for every integer $a \geq 3$.

(b) Show that there exist only finitely many Pythagorean triples (a, b, c) for each integer $a \geq 3$.

(See Eves Problem Study 3.6 and discussion on pp. 81-82)

Problem 34. Given line segment $\overline{AB}$ of length 11 and $\overline{CD}$ of length 3, construct a point C on $\overline{AB}$ such that $|CB| = x$, where x is a solution to the quadratic equation

$$x^2 - 11x + 9 = 0.$$

State the exact value of x. (See Eves Problem Study 3.10a and discussion on pp. 88-89)

Problem 35. Given a two points A and B, construct a point Z such that $\angle BAZ = \angle ABZ = 75°$.

Problem 36. Compute the area of a regular pentagon inscribed in a unit circle.

Problem 37. Draw neatly with straightedge and compass, describing each step.

(a) Given two points, construct an angle of 45°.

(b) Trisect the 45° angle.

(c) Does this show that all angles can by trisected?

Problem 38. Using straightedge and compass, construct an angle of 54°. Describe each step, discussing why your construction is effective.

Problem 39. Compute the volume of a regular icosahedron inscribed in a sphere of radius 1.

Problem 40. (Fibonacci)
Recall that the Fibonacci sequence (F_n) is defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_n + F_{n+1}$, and that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$.

Let $b \in \mathbb{R}$ with $b \geq 1$ and define a sequence (G_n) by $G_1 = 1$, $G_2 = 1$, and $G_{n+2} = G_n + bG_{n+1}$.

Let $c \in \mathbb{R}$ with $c \geq \phi$. Find b such that $\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = c$.

Problem 41. (Tartaglia)
Recall that Tartaglia viewed the cube x^3 as $(t-u)^3$ to find solutions to cubic equations.

Let $f(x) = x^3 + 3x^2 + 6x - 8$. Find the real zero of f using Tartaglia's cube plus cosa method.

Problem 42. (Descartes)
Recall that Descartes used the concept of expanding circles and the ability to compute the number of real solutions to quadratic equations to find tangents.

Find the distance between the curve $x = y^2$ and the point $(3, 0)$ using Descartes' discriminant method.

Problem 43. (Napier)
Recall that Napier desired to find a function to convert multiplication into addition. We may use techniques of Calculus unavailable to him to see that he had very little choice. The modern definition is

$$\log x = \int_1^x \frac{dt}{t} \quad \text{and} \quad \log_b(x) = \frac{\log x}{\log b}.$$

Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function which is not constantly zero and satisfies

$$f(ab) = f(a) + f(b) \quad \text{for all } a, b \in (0, \infty).$$

Show that there exists $b \in \mathbb{R}$ such that $f(x) = \log_b(x)$.

Problem 44. Let $x = \frac{271}{200}$ (expressed in decimal). Find the base sixty radix expansion of x. (Hint: first multiply the numerator and denominator by some number n, then convert the numerator to base six. If you choose n wisely, you will now be almost done.)

Problem 45. Given two points A and B, construct a point C so that $\triangle ABC$ has angles $30°$, $60°$, and $90°$.

Problem 46. Let $m = 52$, $n = 77$, $a = 5$, and $b = 7$.

(a) Find $x, y \in \mathbb{Z}$ such that $mx + ny = 1$.

(b) Find $c \in \mathbb{Z}$ with $0 \leq c < mn$ such that $c \equiv a \pmod{m}$ and $c \equiv b \pmod{n}$.

Problem 47. Consider $\mathbb{Z}_{31} = \{0, 1, \ldots, 30\}$ (the bar notation is understood).

(a) Find $a \in \mathbb{Z}_{31}$ such that $2a = 1$.

(b) Find $c, d \in \mathbb{Z}_{31}$ such that $c^2 = d^2 = 5$.

(c) Let $f(x) = x^2 - x - 1$. Find $m, n \in \mathbb{Z}_{31}$ which are distinct solutions to $f(x) = 0$.

Problem 48. Regarding power series:

(a) Write the Taylor series for e^x.

(b) Use the Taylor series of e^x to find the Taylor series of e^{x^2}.

(c) Use the first four terms of the Taylor series of e^{x^2} to estimate $\int_0^2 e^{x^2}\, dx$.