Math 3063 Calculus I — Prof. Paul Bailey — Project 3 — December 2, 2008 — Name:

Due Tuesday, December 9, 2008.

Write all solutions neatly, in complete sentences. The statement of the problem should always be copied onto a blank sheet of $8\frac{1}{2} \times 11$ computer paper, followed by the solution. Staple this sheet to the front of your solutions.

Problem 1. Let $f(x) = 25 - x^2$. For each $x \in (0, 5)$, consider the triangle bounded by the x-axis, the y-axis, and the line tangent to the graph of f at $(x, f(x))$. Find the minimum area of such a triangle.

Problem 2. Let $f(x) = x^2 + 4$ and let $g(x) = -f(x)$. There is a unique line $y = mx$, with $m > 0$, which is tangent to the graphs of f and g. Find m.

Problem 3. Let $f(x) = cx - x^3$, where $c \in \mathbb{R}$ is positive. Then there exist $a, b \in \mathbb{R}$ with $a < b$ such that f has a local minimum at $x = a$ and a local maximum at $x = b$.

Let m be the slope of the line through $(a, f(a))$ and $(b, f(b))$. Find c such that $m = 1$.

Problem 4. A polynomial is *monic* if its leading coefficient is 1. Let f be a monic fourth degree polynomial with inflection points $(\pm 1, 1)$ and a critical point at $x = 0$.

(a) Find f.

(b) Find all intercepts, extreme points, and inflection points of f.

(c) Sketch the graph of f.

Problem 5. Let $f(x) = x^3 - x^4$. There is a unique line L which is tangent to the graph of f at exactly two points. Let $a, b \in \mathbb{R}$ with $a < b$ such that $P = (a, f(a))$ and $Q = (b, f(b))$ are the points of tangency. Let m be the slope of the line.

We wish to find P and Q, and our intuition tells us that if we subtract the tangent line, we will create a polynomial whose local maximum points both lie on the x-axis. If the x-coordinate of the local minimum is halfway between the local maxima, we may shift the graph the appropriate amount so that it is be symmetric with respect to the y-axis, making the problem tractable.

Find P and Q as follows.

(a) Sketch the graph $f(x)$, and draw the points P and Q and the line L.

(b) By MVT, there exists $c \in (a, b)$ such that $f'(c) = m$. Let $g(x) = f(x+c) - mx$ be the function obtained by shifting f left by c, and subtracting a line of slope m. Sketch the graph of g.

(c) Find c and m such that g is an even function. Since g is a degree four polynomial, it is even exactly if the coefficients of x^3 and x are zero.

(d) Show that $f'(c) = m$.

(e) Use the results of **(c)** to find a and b.