Calculus I (Math 1525)
Midterm Exam # 2
SOLUTIONS

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Problem 1. (True/False)

Let $f : [a, b] \to \mathbb{R}$ be a function such that $f(a) = 2$ and $f(b) = 5$. Then the range of f contains $[2, 5]$.

Answer. If f is continuous, then this is always true as a consequence of the Intermediate Value Theorem. Since it is not given that f is continuous,
(U) Cannot be determined from the information given (depends on f). □

Let $f(x) = x^3 - 3x^2 + 3x - 1$ and let $a < 1$. Then $f'(a) > 0$.

Answer. Note that, by the Binomial Theorem, $f(x) = (x - 1)^3$. So, f is just x^3 shifted right 1, and as such, we know it is always increasing (although it has one horizontal tangent at $x = 1$). Thus $f'(a) > 0$ unless $a = 1$, so
(T) Always True □

Let f and g be differentiable on an open interval I.
If f is concave up on I and g is concave up on I, then $f + g$ is concave up on I.

Answer. Since f and g are differentiable, and concave up, $f''(x) > 0$ and $g''(x) > 0$ for all $x \in I$. Thus $(f + g)''(x) = f''(x) + g''(x) > 0$ for all $x \in I$, and this statement is
(T) Always True □

Let f and g be differentiable on an open interval I.
If f is concave up on I and g is concave down on I, then fg is concave up on I.

Answer. We know that $f'' > 0$ and $g'' < 0$ on I. But we compute, using the product rule, that

$$(fg)'' = f''g + 2f'g' + fg'',$$

whose sign is not readily determined from the given information.

We consider two examples.

If $f(x) = x^2$ and $g(x) = -x^2$, then f is concave up on $\mathbb{R}$ and g is concave down on $\mathbb{R}$, but $(fg)(x) = -x^4$ is concave down on $\mathbb{R}$; perhaps this is what we expect.

If $f(x) = x^2$ and $g(x) = 6 - x^2$, then f is concave up on $\mathbb{R}$ and g is concave down on $\mathbb{R}$. Let $h(x) = (fg)(x) = x^2(6 - x^2) = 6x^2 - x^4$. Then $h'(x) = 12x - 4x^3$ and $h''(x) = 12 - 12x^2$, so $h''(x) = 0$ implies that $x = \pm 1$. On the interval $(-1, 1)$, h is concave up. We conclude that the truth of this statement
(U) Cannot be determined from the information given (depends on f and g). □

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant polynomial with a local max at $x = a$ and a local min at $x = b$.
Then f has an inflection point $x = c$ for some $c \in (a, b)$.

Answer. Apply IVT to f' to see that this statement is
(T) Always True □

Problem 2. (Derivatives)
Compute $\frac{dy}{dx}$ and simplify.

(a) $y = x^5 - 4x^3 + 2x - 1$
By the Power Rule and the fact that differentiation is linear,

$$\frac{dy}{dx} = 5x^4 - 12x^2 + 2.$$

(b) $y = \dfrac{x^3 + \sin x}{x^2 + \cos x}$
By the Quotient Rule,

$$\frac{dy}{dx} = \frac{(3x^2 + \cos x)(x^2 + \cos x) - (x^3 + \sin x)(2x - \sin x)}{(x^2 + \cos x)^2}.$$

(c) $y = \tan(\sqrt{x^3 + 2x^2 + 3})$
By the Chain Rule,

$$\frac{dy}{dx} = \sec^2(\sqrt{x^3 + 2x^2 + 3})\left(\frac{1}{2\sqrt{x^3 + 2x^2 + 3}}\right)(3x^2 + 4x).$$

(d) $\tan y = x$
Implicity differentiate to get

$$(\sec^2 y)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

Problem 3. (Maximization)
Let

$$f(x) = 9 - x^2.$$

Consider a rectangle whose base is on the x-axis and whose other two vertices are above the x-axis on the graph of f. Let x be half of its base.

(a) Find the area of the rectangle as a function of x.

(b) Find the maximum area of such a rectangle.

Solution. The graph of f is a concave down parabola whose vertex is at $(0, 9)$ and whose x-intercepts are $(\pm 3, 0)$. The rectangle is therein inscribed. Clearly its base is $2x$ and its height is $9 - x^2$. So

$$A(x) = 2x(9 - x^2) = 18x - x^3.$$

To maximize the area, we take the derivative $A'(x) = 18 - 3x^2$ and set it to zero to obtain $3x^2 = 18$, so $x = \pm\sqrt{6}$. Since x must be positive, we plug $\sqrt{6}$ into A to get that the maximum area is

$$A(\sqrt{6}) = 18\sqrt{6} - 6\sqrt{6} = 12\sqrt{6}.$$

□

Problem 4. (Curve Sketching)
Find the extreme points of the function

$$f(x) = x^4 - x^3 - 2x^2 + 3x,$$

and use these to sketch its graph.

Solution. To find extreme points, we take the first derivative and set it to zero. To solve this polynomial equation, we will factor the derivative. Now

$$f'(x) = 4x^3 - 3x^2 - 4x + 3 = x^2(4x - 3) - (4x - 3) = (x^2 - 1)(4x - 3) = (x + 1)(x - 1)(4x - 3).$$

Thus $f'(x) = 0$ for $x = -1$, 1, and $\frac{3}{4}$. Since $f(x) > 0$ for x for large x, it is clear that $x = \pm 1$ give local minima and $x = \frac{3}{4}$ gives a local maximum. To find the exact points on the curve, plug these numbers into f to obtain $f(-1) = -3$, $f(1) = 1$, and $f(\frac{3}{4}) = \frac{261}{256}$.

Thus we see that $(-1, -3)$ and $(1, 1)$ are local minima, and $(\frac{3}{4}, \frac{261}{256})$ is a local maximum. Moreover, it is clear that $f(0) = 0$. Plotting this information gives an adequate graph. □

Problem 5. (Analytic Geometry)
Find the slope-intercept form of the equation of a line which passes through the point $(5, 0)$ and is tangent to the graph of the function $f(x) = 16 - x^2$.
(Hint: call the point of tangency $(a, f(a))$ and sketch a picture. Then find a.)

Solution. We have $f'(x) = -2x$.

Let $(a, f(a))$ denote the point of tangency. Note $f9a) = 16 - a^2$. The slope of the tangent line is $-2a$, and $(5, 0)$ is on the line, so its equation is

$$y = -2a(x - a) + f(a) = (-2ax + 2a^2) + (16 - a^2) = -2ax + (a^2 + 16).$$

Since $(5, 0)$ is on the line, $0 = -2a(5) + a^2 + 16$, or $a^2 - 10a + 16 = 0$, so $(a - 2)(a - 8) = 0$, so either $a = 2$ or $a = 8$. These are both acceptable solutions (as one sees from the picture). Thus the possible lines are

$$y = -4x + 20 \quad \text{or} \quad y = -16x + 80.$$

□

Problem 6. (Bonus - Diophantine Geometry)
A *rational curve* is the locus of a polynomial equation in two variables whose coefficients are rational numbers. A *rational point* on a curve is a solution whose coordinates are rational numbers.

Diophantus (Alexandria, 2^{nd} century A.D.) discovered that, given one rational point on a rational curve whose tangent line has a rational slope, he could find another rational point by following the tangent line.

Consider the curve given by the equation

$$y^2 = x^3 - 3x^2 + 3x + 1.$$

(a) Verify that $(0, 1)$ is a rational point on the curve.

(b) Use implicit differentiation to find $\frac{dy}{dx}$.

(c) Compute the line tangent to the curve at the point $(0, 1)$.

(d) The tangent line intersects the curve in another rational point. Find this point.

Solution. Since $1 = 0^3 - 3(0^2) + 3(0) + 1$, the point $(0, 1)$ is on the curve. Now $2y\frac{dy}{dx} = 3x^2 - 6x + 3$, so

$$\frac{dy}{dx} = \frac{3}{2} \cdot \frac{x^2 - 2x + 1}{y}.$$

Plugging in $(0, 1)$ gives

$$\frac{dy}{dx}\,|_{(0,1)} = \frac{3}{2} \cdot \frac{0^2 - 2(0) + 1}{1} = \frac{3}{2},$$

so the slope of the line tangent to the curve at $(0, 1)$ is $\frac{3}{2}$. Thus the equation of the tangent line is $y = \frac{3}{2}x + 1$.

We intersect the tangent line with the curve to find another rational point. Plugging in $\frac{3}{2}x + 1$ for y in the equation of the curve gives

$$\begin{aligned}
\left(\frac{3}{2}x + 1\right)^2 = x^3 - 3x^2 + 3x + 1 &\Rightarrow \frac{9}{4}x^2 + 3x + 1 = x^3 - 3x^2 + 3x + 1 \\
&\Rightarrow \frac{9}{4}x^2 = x^3 - 3x^2 \\
&\Rightarrow x^3 - \frac{21}{4}x^2 = 0 \\
&\Rightarrow x^2\left(x - \frac{21}{4}x\right) = 0 \\
&\Rightarrow x = 0 \text{ or } x = \frac{21}{4}.
\end{aligned}$$

We started at $x = 0$, so the solution we seek is $x = \frac{21}{4}$. The corresponding y coordinate is given by the line as

$$y = \frac{3}{2}\left(\frac{21}{4}\right) + 1 = \frac{71}{8}.$$

Thus $\left(\frac{21}{4}, \frac{71}{8}\right)$ is another rational point on the curve. $\square$