History of Mathematics (Math 4123)
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Solutions to Chapter Six Homework

Problem 1. (6.7 Apollonius on Tangencies) In his lost treatise on *Tangencies*, Apollonius considered the problem of drawing a circle tangent to three circles, including degenerate forms of a circle including a point or a line.

(a) Find the number of cases, depending on whether we have points, lines, or circles, and the maximum number of solutions in each case.

(b) Given points A and B and line L, find all circles through the points and tangent to the line.

(c) Reduce the case of two lines and a point to the case of part **(b)**.

We will use the following geometric lemmas.

Lemma 1. *Let T be a point on a circle C with center D, and let L be a line through T. Then L is tangent to C if and only if the line through D and T is perpendicular to L.*

Lemma 2. *Let A, B, and C be distinct points. Then there exists a unique circle through A, B, and C whose center is the intersection of the perpendicular bisectors of the line segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$.*

Proof. The set of points equally distant between two given points is the line perpendicular to the line through the given points which passes through their midpoint. Thus the center of the circle lies on this line, for each pair of points. $\square$

Lemma 3. (Central Angle Theorem)
Let A, B, and C be points on a circle with center D. Then $\angle ADB = 2\angle ACB$.

Proof. The triangles $\triangle ADB$, $\triangle BDC$, and $\triangle CDA$ are isosceles. Let $y = ACB$ and $2x = \angle ADB$. Now

$$\begin{aligned}
180^\circ &= \angle CAB + \angle CBA + \angle ACB \quad \text{(because the sum of the angle is } 180^\circ\text{)} \\
&= (\angle CAD + \angle DAB) + (\angle CBD + \angle DBA) + (\angle ACD + \angle DCB) \quad \text{(breaking up the angles)} \\
&= (\angle DAB + \angle DBA) + (\angle CBD + \angle CAD) + (\angle ACD + \angle DCB) \quad \text{(reassociating)} \\
&= (\angle DAB + \angle DBA) + 2(\angle ACD + \angle DCB) \quad \text{(because } \angle CBD = \angle ACD \text{ and } \angle CAD = \angle DCB\text{)} \\
&= (180^\circ - 2x) + 2y.
\end{aligned}$$

Thus $2y - 2x$, so $y = x$. $\square$

Lemma 4. *Let A and B be points and let M be the line through A and B. Let L be a line which is not parallel to M and let S be the point of intersection of L and M. Let T be a point on L. Then L is tangent to the circle through A, B, and T if and only if $\angle STA = \angle SBT$.*

Proof. Let $x = \angle STA$. Let N be the line through T and D. Then

$$C \text{ is tangent to } L \Leftrightarrow L \perp N \quad \Leftrightarrow \angle STD = 90^\circ \Leftrightarrow \angle ATD = 90^\circ - x \quad \Leftrightarrow \angle ADT = 2x \Leftrightarrow \angle ABT = x.$$

$\square$

Lemma 5. *Let A and B be points and let M be the line through A and B. Let L be a line which is not parallel to M and let S be the point of intersection of L and M. Let T be a point on L. Then L is tangent to the circle through A, B, and T if and only if*

$$(SA)(SB) = (ST)^2.$$

Proof. By the previous lemma, L is tangent to the circle if and only if $\triangle AST \sim \triangle TSB$, which is true if and only if

$$\frac{AS}{ST} = \frac{ST}{SB}.$$

The result follows. □

Solution.

(a) Letting p mean point, l mean line, and c mean circle, there are ten cases:

$$ppp, ppl, pll, lll, ppc, pcc, ccc, llc, lcc, plc.$$

Type ppp has a unique solution; each of the other types has two solution in general.

Two find the unique circle through three points, we take the center to be the intersection of the perpendicular bisectors of the line segments between the points. Thus, the other problems may be reduced to finding the points of tangency on the given lines or circles.

(b) If the line through A and B is parallel to C, then there is a unique solution. The point of tangency on C is obtained by intersecting C with the perpendicular bisector of $\overline{AB}$.

Otherwise, let S be the intersection of the line through A and B and the line C. Let T be a point on C such that $(ST)^2 = (SA)(SB)$. Then T is a point of tangency. There are two solutions (one on either side of S). The proof that this is so follows.

(c) Let A be a point and let L and M be lines.

Let N be the line which bisects the angle between L and M. If L and M happen to be parallel, let N be the midline. Reflect A through this line to obtain a point B. This reduces this case to the previous case, except if A is on N.

If A is on N, construct the line through A perpendicular to N, and let E be the point of intersection. Bisect the angle at E and let D be the point of intersection of the bisecting line with N. The D is the center of the tangent circle (their are two solutions). To see this, construct the line through D perpendicular to L and intersect it with L at point T. Then $\triangle AED \cong \triangle TED$ by AAS, so $AD = TD$; this is the radius of the circle. □

Problem 2. (6.10 Stereographic Projection) See Wikipedia on this topic. The formulae for stereographic projection are given there, together with the algebraic proof that circles are mapped to circles.

Problem 3. (6.15 Diophantus)

(a) About all we know of Diophantus' personal life is contained in the following summary of an epitaph given in the *Greek Anthology*: "Diophantus passed $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth, and $\frac{1}{7}$ more as a bachelor. Five years after his marriage was born a son who died 4 years before his father, at $\frac{1}{2}$ his father's [final] age." How old was Diophantus when he died?

(b) Solve the following problem, which appears in Diophantus' *Arithmetica* (Problem 17, Book I): Find 4 numbers, the sum of every arrangement 3 at a time being given; say 22, 24, 27, 20.

(c) Solve the following problem, also found in the *Arithmetic* (Problem 16, Book VI): In the right triangle ABC, right angled at C, AD bisects angle A. Find the set of smallest integers for AB, AD, AC, BD, DC such that $DC : CA : AD = 3 : 4 : 5$.

Solution.

(a) Let x be the age of Diophantus at his death, and let y be the number of years he lived after his marriage. Thus $y = x - (\frac{1}{6} + \frac{1}{12} + \frac{1}{7})x$. That is, $y = \frac{17}{28}x$. Also, the final age of his son is $\frac{1}{2}x = (y - 5 - 4) = \frac{17}{28}x - 9$. Solving for x gives $x = 84$.

(b) We could create a system of four linear equations in four variables and solve it using matrix techniques; it is less computationally intense to proceed as follows.

Let r, s, t, u be the given numbers and let a, b, c, d be the unknown numbers. Let v be the sum of the given numbers, which we can compute up front, and let e be the sum of the unknown numbers. Without loss of generality, assign

$$a = e - r, b = e - s, c = e - t, d = e - u.$$

Adding these equations gives $e = 4e - v$, so $3e = v$. So, $e = \frac{v}{3}$. From this, produce a, b, c, d.

For example, if $r = 22$, $s = 24$, $t = 27$, and $u = 20$, we have $v = 93$, so $e = 31$. Thus

$$a = 9, b = 7, c = 4, d = 11.$$

(c) Let $\theta = \angle CAD$, so that $2\theta = \angle CAB$. We want $\tan\theta = \frac{3}{4}$. Then

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{24}{7}.$$

Let $CD = 3x$ so that $CA = 4x$. Let $y = CB$. Then

$$\tan 2\theta = \frac{y}{4x} = \frac{24}{7}.$$

We see that x must be a multiple of 7; trying $x = 7$, we have $y = 96$, and

$$(AB)^2 = 28^2 + 96^2 = 4^2(7^2 + 24^2) = 4^2(25^2).$$

Thus $x = 7$ produces an integer hypotenuse, and

$$AB = 100, AD = 35, AC = 28, BD = 75, DC = 21.$$

$\square$