Math 3063 Calculus I Project 1 Solutions
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Definition 1. Let $D \subset \mathbb{R}$ be an interval and let $f : D \to \mathbb{R}$. Let $a \in D$. We say that f is *continuous* at a if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in D$, we have

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

Problem 1. Let

$$f : \mathbb{R} \to \mathbb{R} \quad \text{be given by} \quad f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}; \\ 0 & \text{otherwise.} \end{cases}$$

Apply the definition to show that f is continuous at 0.

Solution. If $x \in \mathbb{Q}$, then $f(x) = x$, so $|f(x)| = |x|$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x) = 0$, so $|f(x)| = |0| = 0 \leq |x|$. Thus, for all $x \in \mathbb{R}$, we have $|f(x)| \leq |x|$.

Let $\epsilon > 0$. To show that f is continuous at 0, we wish to find $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon,$$

where $a = 0$.

Set $\delta = \epsilon$; we now show that this δ has the desired property. Suppose $|x - a| < \delta$. Since $a = 0$ and $f(0) = 0$, we have

$$|f(x) - f(a)| = |f(x) - f(0)| = |f(x) - 0| = |f(x)| \leq |x| = |x - 0| = |x - a| < \delta = \epsilon;$$

that is, $|x - a| < \delta \Rightarrow |f(x) = f(a)| < \epsilon$. □

Theorem 1. Intermediate Value Theorem (IVT)
Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Problem 2. Let $f : [0, 1] \to [0, 1]$ be continuous.
Apply the Intermediate Value Theorem to show that there exists $c \in [0, 1]$ such that $f(c) = c$.

Solution. What we know about f is that its graph is a continuous curve in the square $[0, 1] \times [0, 1]$ which touches the y-axis and the line $x = 1$. The line segment from the origin to the point $(1, 1)$ must cross this curve. This line segment is the portion of the graph of $y = x$ which resides in this square. We use this picture to set up the Intermediate Value Theorem.

Let $g(x) = x - f(x)$. Then g is the difference of continuous functions, so g is continuous.

Since $0 \leq f(0) \leq 1$, we have $g(0) = 0 - f(0) \leq 0$.

Since $0 \leq f(1) \leq 1$, we have $g(1) = 1 - f(1) \geq 0$.

If $g(0) = 0$, let $c = 0$.

Otherwise, if $g(1) = 0$, let $c = 1$.

Otherwise, we have $g(0) < 0$ and $g(1) > 0$, so by the Intermediate Value Theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$.

In all three cases, $g(c) = 0$, so $g(c) = c - f(c) = 0$, so $f(c) = c$. □

Definition 2. Let $A \subset \mathbb{R}$.
We say that A is *globally discrete* if

$$\text{there exists } \epsilon > 0 \text{ such that for every } a \in A, (a-\epsilon, a+\epsilon) \cap A = \{a\}.$$

We say that A is *locally discrete* if

$$\text{for every } a \in A \text{ there exists } \epsilon > 0 \text{ such that } (a-\epsilon, a+\epsilon) \cap A = \{a\}.$$

We say that A is *indiscrete* if A is not locally discrete.
We say that A is *bounded* if there exists $a, b \in \mathbb{R}$ such that $A \subset [a, b]$.

Problem 3. Which of the following sets are 1) bounded; 2) globally discrete; 3) locally discrete; 4) indiscrete? Describe why.

(a) $A = \mathbb{Q}$

(b) $B = \mathbb{Z}$

(c) $C = [2, 5]$

(d) $D = [2, 5] \cap \mathbb{Q}$

(e) $E = [2, 5] \cap \mathbb{Z}$

(f) $F = \{x \in \mathbb{R} \mid x = \frac{1}{n} \text{ for some } n \in \mathbb{N}\}$

Solution. First we note that between any two real numbers, there exist both a rational and an irrational number.

Clearly, A and B are unbounded, and C, D, and E are bounded. Also, $F \subset [0, 1]$, so F is bounded.

(a) A is indiscrete; if $a \in \mathbb{Q}$ and $\epsilon > 0$, there exists a rational number b between a and $a+\epsilon$, so $(a-\epsilon, a+\epsilon)$ includes b.

(b) B is globally discrete; if $a \in \mathbb{Z}$ and $\epsilon = \frac{1}{2}$, then then only integer in $(a - \frac{1}{2}, a + \frac{1}{2})$ is a.

(c) C is indiscrete, similarly to A

(d) D is indiscrete, similarly to A

(e) E is globally discrete, similarly to B

(f) F is locally discrete but not globally discrete. To see this, let $n \in \mathbb{N}$. The two points in F which are closest to $\frac{1}{n}$ are $\frac{1}{n+1}$ and $\frac{1}{n-1}$. The closer of these is $\frac{1}{n+1}$. If we let

$$\epsilon = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n},$$

then $(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$ contains only $\frac{1}{n}$ from the set F.

However, F is not globally discrete, because as n gets larger, $\frac{1}{n}$ gets closer to zero, and $\frac{1}{n^2+n}$ gets smaller and smaller. No particular ϵ will work for all n.

To say this precisely, we let $\epsilon > 0$ be arbitrary, and see that this epsilon does not work for all n. In particular, let n be so large $n^2 + n > \frac{1}{\epsilon}$; then

$$\frac{1}{n} - \epsilon < \frac{1}{n+1} < \frac{1}{n},$$

so

$$\frac{1}{n+1} \in \left(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon\right).$$

$\square$

Problem 4. Answer the following questions. Explain your reasoning.

(a) Is every finite set bounded?

(b) Is every globally discrete set locally discrete?

(c) Is every finite set globally discrete?

(d) Is every infinite set indiscrete?

(e) Is every bounded infinite set indiscrete?

Solution.

(a) Yes, every finite set is bounded. Let A be finite. Then A has a minimum and a maximum. Let $a = \min(A)$ and $b = \max(A)$. Then $A \subset [a, b]$.

(b) Yes, every globally discrete set is locally discrete. This is clear; in words, the same epsilon separates every point in the set from every other point, so every point can be separated.

(c) Yes, every finite set is globally discrete. Let A be a finite set. Let

$$D = \{|a_1 - a_2| \mid a_1, a_2 \in A\}.$$

In words, D is the set of all distances between two points in A. Since A is finite, so is D. Thus D has a minimum. Let $\epsilon = \min D$.

(d) No, not every infinite set is indiscrete. An example of an infinite discrete set is $\mathbb{Z}$, which is globally discrete. Another example is Problem 3 Set F, which is infinite yet locally discrete.

(e) No, not every bounded infinite set is indiscrete, for example, Problem 3 Set F is locally discrete.

It is true that a bounded infinite set cannot be globally discrete. To see this, consider a set bounded set $A \subset [a, b]$. Dividing the interval in half, one of the halves must contain an infinite number of elements from A. Take that half and again divide it in half, and repeat this process. You will find that an infinite number of points reside in a subinterval which is as small as you wish. This is sometimes called the *pigeonhole principle*; if you try to put an infinite number of pigeons in a finite number of holes, one of the hole must contain an infinite number of pigeons. $\square$