MATH 3063 CALCULUS I — PROF. PAUL BAILEY — Project 2 Solutions — November 24, 2008 — Name:

Let $P, Q \in \mathbb{R}^2$. Denote the distance between P and Q by $d(P, Q)$.

Let $A, B \subset \mathbb{R}^2$ be nonempty. The *distance* between A and B, denoted $d(A, B)$, is the largest nonnegative real number such that

$$d(A, B) \leq d(P, Q) \text{ for all } P \in A \text{ and } Q \in B.$$

Example 1. Let $A = \{(1, 2)\}$ and $B = \{(5, 1)\}$. Then $d(A, B) = \sqrt{(5-1)^2 + (1-2)^2} = \sqrt{17}$.

Example 2. Let $A = \{(0, 4)\}$ and let B denote the x-axis. Then $d(A, B) = 4$.

Example 3. Let $A = \{(x, y) \in \mathbb{R}^2 \mid y = 2x + 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid y = 3x - 5\}$. Since A and B are nonparallel lines, they intersect, so $d(A, B) = 0$.

Example 4. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid (x - 6)^2 + y^2 = 4\}$. Then $d(A, B) = 3$.

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $P \in \mathbb{R}^2$ be a point not on the graph of f. Let $A = \{P\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$. Let Q be point on the graph of f such that $d(A, B) = d(P, Q)$. Show that the line through Q tangent to the graph of f is perpendicular to the line through P and Q.

Solution. Let $P = (a, b)$ and $Q = (x, f(x))$. The distance between them is

$$\rho(x) = \sqrt{(x - a)^2 + (f(x) - b)^2}.$$

To minimize ρ, we set $\rho'(x) = 0$ and solve to x. Use the chain rule while taking the derivative of ρ to obtain

$$\rho'(x) = \frac{(x - a) + (f(x) - b)f'(x)}{\sqrt{(x - a)^2 + (f(x) - b)^2}}.$$

Since the denominator does not effect whether $\rho'(x)$ is zero, we set the numerator to zero and obtain

$$(x - a) + (f(x) - b)f'(x) = 0 \quad \Rightarrow \quad f'(x) = -\frac{x - a}{f(x) - b},$$

which is the negative reciprocal of the slope of the line through P and Q. Thus these line are perpendicular. □

Problem 2. Let

$$A = \{(x, y) \mid y = 3x - 5\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 \mid y = 3x + 3\}.$$

Find $d(A, B)$.

Solution. We note that A and B are parallel lines, and as such, we can choose any point in B and find its minimum distance to A. The point $(0, 3)$ is in B. The line through $(0, 3)$ and perpendicular to B is $y = -\frac{1}{3}x + 3$. Intersecting this with A produces

$$3x - 5 = -\frac{1}{3}x + 3 \Rightarrow (3 + \frac{1}{3})x = 8 \Rightarrow x = \frac{24}{10} = \frac{12}{5} \Rightarrow y = \frac{36}{5} - 5 = \frac{11}{5}.$$

So the closest point in A to $(0, 3)$ is $(\frac{12}{5}, \frac{11}{5})$, and the distance between them is

$$d(A, B) = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{11}{5} - 3\right)^2} = \sqrt{\frac{144}{25} + \frac{16}{25}} = \frac{4\sqrt{10}}{5}.$$

□

Problem 3. Let

$$A = \{(x, y) \mid y = (x-2)^2 + (y-3)^2 = 4\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 \mid (x+5)^2 + (y+5)^2 = 9\}.$$

Find $d(A, B)$.

Solution. In this case, A and B are circles which do not intersect. The distance between them is realized as the distance between the point on A and the point found by intersecting the circles with a line through their centers. We do not need to calculate these points, however, since it is clear that the distance between the circles is simply the distance between their centers, minus their radii.

The center of A is $(2, 3)$ and the center of B is $(-5, -5)$; the distance between these points is $\sqrt{7^2 + 8^2} = \sqrt{113}$. The radius of of A is 2 and the radius of B is 3, so

$$d(A, B) = \sqrt{113} - 5.$$

□

Problem 4. Let $a \in \mathbb{R}$. Let

$$A = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\} \quad \text{and} \quad B = \{(0, b)\}.$$

The distance between A and B is a function of b.
Find $\rho(b) = d(A, B)$.

Solution. Let $P = (0, b)$ and $Q = (x, y) \in A$. Then $y = x^2$. Let $f(x) = (d(P, Q))^2$; then the distance from A to B occurs at a value for x where $f(x)$ is minimal. To find this, derive f, set $f'(x) = 0$, and solve for x.

We have

$$f(x) = (x - 0)^2 + (y - b)^2 = x^2 + (x^2 - b)^2;$$

thus

$$f'(x) = 2x + 2(x^2 - b)(2x) = 2x(2x^2 - 2b + 1).$$

Thus $f'(x) = 0$ implies $x = 0$ or $2x^2 - 2b + 1 = 0$. Solving the second equation gives $x = \pm\sqrt{b - \frac{1}{2}}$.

If $b < \frac{1}{2}$, then $2x^2 - 2b + 1 = 0$ has no real solution; in this case, $x = 0$ represents a minimum, and the closest point on the parabola to $(0, b)$ is the vertex at the origin, so $\rho(b) = b$.

If $b = \frac{1}{2}$, $\sqrt{b - \frac{1}{2}} = \frac{1}{2}$, and the vertex is still the closest point to $(0, b)$.

If $b > \frac{1}{2}$, the two solutions $x = \pm\sqrt{b - \frac{1}{2}}$ represent two points on opposite sides of the parabola, equally distance from $(0, b)$, giving local minima to the distance squared function f.

Set $c = \sqrt{b - \frac{1}{2}}$; then

$$f(c) = c^2 + (c^2 - b)^2 = (b - \frac{1}{2}) + (b - \frac{1}{2} - b)^2 = b - \frac{1}{4}.$$

So the distance from $(0, b)$ to (c, c^2) is $\sqrt{f(c)} = \sqrt{b - \frac{1}{4}}$. Note that, for $b > \frac{1}{2}$, we have $\sqrt{b - \frac{1}{4}} < b$, so this is less than the distance to the origin; the distance to the vertex is a local maximum.

We conclude that

$$\rho(b) = \begin{cases} b & \text{if } b \leq \frac{1}{2}; \\ \sqrt{b - \frac{1}{4}} & \text{if } b \geq \frac{1}{2}. \end{cases}$$

As an aside, we note that the focus of the parabola A is at $(0, \frac{1}{4})$; the vertex is the closest point to a point on the axis of symmetry until the distance from that point to the focus is exceeds the distance from the focus to the vertex. □

The solution to the last problem is simplified by the following lemma.

Lemma 1. *Let $A : y = mx + b$ and $B : y = mx$ be the equations of parallel lines, with $b \geq 0$. Then the distance between the lines is*

$$d(A, B) = \frac{b}{\sqrt{m^2 + 1}}.$$

Proof. Note $(0, 0)$ is on the line B. The line through $(0, 0)$ perpendicular to B is $C : y = -\frac{1}{m}x$. Compute the intersection of A with C:

$$mx + b = -\frac{1}{m}x \quad \Rightarrow \quad \frac{m^2 + 1}{m}x = -b \quad \Rightarrow \quad x = -\frac{bm}{m^2 + 1}.$$

The y-coordinate of the point of intersection is $y = \dfrac{b}{m^2 + 1}$. The distance from $(0, 0)$ to this point of intersection is

$$\sqrt{\left(\frac{-bm}{m^2 + 1}\right)^2 + \left(\frac{b}{m^2 + 1}\right)^2} = \sqrt{\frac{b^2m^2 + b^2}{(m^2 + 1)^2}} = \sqrt{\frac{b^2(m^2 + 1)}{(m^2 + 1)^2}} = \frac{b}{\sqrt{m^2 + 1}}.$$

$\square$

Problem 5. Let $m \in \mathbb{R}$. Let

$$A = \{(x, y) \mid y = x^2 + 9\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 \mid y = mx\}.$$

The distance between A and B is a function of the slope m.
Find $\rho(m) = d(A, B)$.

Solution. First we compute the intersection of A and B:

$$x^2 + 9 = mx \quad \Rightarrow \quad x^2 - mx + 9 = 0 \quad \Rightarrow \quad x = \frac{m \pm \sqrt{m^2 - 36}}{2}.$$

If $m = 6$, we have exactly one real solution; this is the case where B is tangent to A. The point of tangency is $(\frac{m}{2}, \frac{m^2}{4} + 9)$. If $m > 6$, there are two real solutions, corresponding to two points of intersection. If $m \geq 6$, the curves intersect, so the distance between them is zero.

Now we consider the case where $m < 6$. Let $f(x) = x^2 + 9$. Then $f'(x) = 2x$. By Problem 1, the point on A closest to B has a tangent parallel to B. Let (a, b) denote this point; then $f'(a) = 2a = m$, so $a = \frac{m}{2}$, and $b = \frac{m^2}{4} + 9 = \frac{m^2+36}{4}$.

The line tangent to f at (a, b) is

$$y = m(x - \frac{m}{2}) + \frac{m^2 + 36}{4} = mx + \frac{36 - m^2}{4}.$$

The distance between this line and B is given by our lemma as $\frac{36-m^2}{4\sqrt{m^2+1}}$.

Therefore

$$\rho(m) = \begin{cases} \dfrac{36 - m^2}{4\sqrt{m^2 + 1}} & \text{if } m \leq 6; \\ 0 & \text{if } m \geq 6. \end{cases}$$

$\square$