MATH 1525 CALCULUS I — PROF. PAUL BAILEY — Project 3 — December 11, 2008 — Name:

Problem 1. Let $f(x) = 25 - x^2$. For each $x \in (0, 5)$, consider the triangle bounded by the x-axis, the y-axis, and the line tangent to the graph of f at $(x, f(x))$. Find the minimum area of such a triangle.

Solution. Let $a \in (0, 5)$. The equation of the tangent line at $(a, f(a))$ is

$$y = f'(a)(x - a) + f(a) = -2a(x - a) + 25 - a^2 = -2ax + a^2 + 25.$$

The vertices of the triangle are the origin, the y-intercept of the tangent line, and the x-intercept of the tangent line. The y-intercept is at $y = a^2 + 25$, and the x-intercept is at $x = \frac{a^2+25}{2a}$.

The base of the right triangle is $b = \frac{a^2+25}{2a}$ and the height is $h = a^2 + 25$, so the area is

$$A(a) = \frac{1}{2}bh = \frac{(a^2 + 25)^2}{4a}.$$

To minimize the area, we set its derivative equal to zero.We have

$$\frac{dA}{da} = \frac{2(a^2 + 25)(2a)(4a) - 4(a^2 + 25)^2}{(4a)^2} = \frac{4a^2(a^2 + 25) - (a^2 + 25)^2}{4a^2}.$$

We factor $(a^2 + 25)$ out of the numerator, set the numerator equal to zero, and solve to get

$$(a^2 + 25)(4a^2 - (a^2 + 25)) = 0 \quad \Rightarrow \quad 3a^2 - 25 = 0 \quad \Rightarrow \quad a = \frac{5}{\sqrt{3}}.$$

The minimum area occurs when $a = \frac{5}{\sqrt{3}}$. The minimum area, then, is

$$A\Big(\frac{5}{\sqrt{3}}\Big) = \frac{100^2\sqrt{3}}{20 \cdot 3^2} = \frac{500\sqrt{3}}{9}.$$

□

Problem 2. Let $f(x) = x^2 + 4$ and let $g(x) = -f(x)$. There is a unique line $y = mx$, with $m > 0$, which is tangent to the graphs of f and g. Find m.

Solution. We have $g(x) = -x^2 - 4$.

Let $(a, f(a))$ be the point of tangency on the graph of f and let $(b, g(b))$ be the point of tangency on the graph of g. Since these points share a tangent line, their derivatives are equal, so $f'(a) = g'(b)$. Thus $2a = -2b$, so $b = -a$. Also, $g(b) = -f(b) = -f(-a) = -f(a)$.

The slope of the line through these points equals the common derivative, so

$$\frac{g(b) - f(a)}{b - a} = \frac{2f(a)}{2a} = \frac{f(a)}{a} = \frac{a^2 + 4}{a} = 2a.$$

Thus $a^2 + 4 = 2a^2$, so $a^2 = 4$, so $a = 2$. Thus $m = \frac{4+4}{2} = 4$. □

Problem 3. Let $f(x) = cx - x^3$, where $c \in \mathbb{R}$ is positive. Then there exist $a, b \in \mathbb{R}$ with $a < b$ such that f has a local minimum at $x = a$ and a local maximum at $x = b$.

Let m be the slope of the line through $(a, f(a))$ and $(b, f(b))$. Find c such that $m = 1$.

Solution. We first note that f is an odd function which is positive on the far left and negative on the far right. Since $f(x) = x(c - x^2) = -x(x - \sqrt{c})(x + \sqrt{c})$, the zeros of f are 0 and $\pm\sqrt{c}$. Thus we more or less know its graph.

The derivative of f is $f'(x) = c - 3x^2$, so f has a maximum at $b = \sqrt{\frac{c}{3}}$ and a minimum at $a = -b$. We see that

$$f(b) = c\sqrt{\frac{c}{3}} - \frac{c}{3}\sqrt{\frac{c}{3}} = \frac{2c}{3}\sqrt{\frac{c}{3}}.$$

Finally,

$$m = \frac{f(b) - f(a)}{b - a} = \frac{2f(b)}{2b} = \frac{f(b)}{b} = \frac{2c}{3}.$$

If $m = 1$, then $c = \frac{3}{2}$. □

Problem 4. A polynomial is *monic* if its leading coefficient is 1. Let f be a monic fourth degree polynomial with inflection points $(\pm 1, 1)$ and a critical point at $x = 0$.

(a) Find f.

(b) Find all intercepts, extreme points, and inflection points of f.

(c) Sketch the graph of f.

Solution. Writing f with general coefficients and taking derivatives, we have

- $f(x) = x^4 + ax^3 + bx^2 + cx + d$
- $f'(x) = 4x^3 + 3ax^2 + 2bx + c$
- $f''(x) = 12x^2 + 6ax + 2b$

Since f has inflection points at $x = \pm 1$, $f''(1) = 0$ and $f''(-1) = 0$. Thus $12 + 6a + 2b = 0$, and $12 - 6a + 2b = 0$. Subtracting these equations gives $12a = 0$, so $a = 0$. Thus $12 + 2b = 0$, so $b = -6$. Since f has a local extremum at $x = 0$, $f'(0) = 0$, so $c = 0$. Finally, since $f(1) = 1$, we have $1 + 0 - 6 + 0 + d = 1$, so $d = 6$.

Now

- $f(x) = x^4 - 6x^2 + 6$
- $f'(x) = 4x^3 - 12x$
- $f''(x) = 12x^2 - 12$

We note that f is an even function, and that $(\pm 1, 1)$ are the only inflection points. Now $f'(x) = 4x(x^2 - 3)$, so f has local minima at $x = \pm\sqrt{3}$ as well as $x = 0$. Since $f(\sqrt{3}) = 9 - 6(3) + 6 = -3$, the local minimum points are $(\pm\sqrt{3}, -3)$, and the local maximum point in $(0, 6)$, which is the y-intercept.

By the quadratic formula, $f(x) = 0$ implies

$$x^2 = \frac{6 \pm \sqrt{36 - 24}}{2} = 3 \pm \sqrt{3}.$$

So the four x-intercepts of f are $(\pm\sqrt{3 \pm \sqrt{3}}, 0)$. □

Problem 5. Let $f(x) = x^3 - x^4$. There is a unique line L which is tangent to the graph of f at exactly two points. Let $a, b \in \mathbb{R}$ with $a < b$ such that $P = (a, f(a))$ and $Q = (b, f(b))$ are the points of tangency. Let m be the slope of the line.

We wish to find P and Q, and our intuition tells us that if we subtract the tangent line, we will create a polynomial whose local maximum points both lie on the x-axis. If the x-coordinate of the local minimum is halfway between the local maxima, we may shift the graph the appropriate amount so that it is be symmetric with respect to the y-axis, making the problem tractable.

Find P and Q as follows.

(a) Sketch the graph $f(x)$, and draw the points P and Q and the line L.

(b) By MVT, there exists $c \in (a, b)$ such that $f'(c) = m$. Let $g(x) = f(x+c) - mx$ be the function obtained by shifting f left by c, and subtracting a line of slope m. Sketch the graph of g.

(c) Find c and m such that g is an even function. Since g is a degree four polynomial, it is even exactly if the coefficients of x^3 and x are zero.

(d) Show that $f'(c) = m$.

(e) Use the results of **(c)** to find a and b.

Solution. We have

$$\begin{aligned} g(x) = f(x+c) - mx &= (x+c)^3 - (x+c)^4 - mx \\ &= (x^3 + 3cx^2 + 3c^2x + c^3) - (x^4 + 4cx^3 + 6c^2x^2 + 4c^3x + c^4) - mx \\ &= -x^4 + (1-4c)x^3 + (3c - 6c^2)x^2 + (3c^2 - 4c^3 - m)x + (c^3 - c^4). \end{aligned}$$

We wish to find c and m such that g is even. This requires $1 - 4c = 0$, so $c = \frac{1}{4}$, and $3c^2 - 4c^3 - m = 0$, so $\frac{3}{16} - \frac{4}{64} - m = 0$, so $m = \frac{1}{8}$. Now $f'(x) = 3x^2 - 4x^3$, and $f'(\frac{1}{4}) = \frac{3}{16} - \frac{4}{64} = \frac{1}{8}$, so $f'(c) = m$.

We have

$$g(x) = -x^4 + (\frac{3}{4} - \frac{6}{16})x^2 - (\frac{1}{64} + \frac{1}{256}) = -x^4 + \frac{3}{8}x^2 + \frac{3}{256}.$$

□

Now $g'(x) = -4x^3 + \frac{3}{4}x$, so $g'(x) = 0$ if $x = 0$ (the local maximum) or $x^2 = \frac{3}{16}$, so $x = \pm\frac{\sqrt{3}}{4}$. Shift these to the left by c to get

$$a = -\frac{\sqrt{3}+1}{4} \quad \text{and} \quad b = \frac{\sqrt{3}-1}{4}.$$