Due Date: Friday, December 5,2008.
Write your solutions neatly on separate pieces of paper and attach this sheet to the front.
Problem 4 may require some ingenuity, but is a fascinating result.

Problem 1. (Fibonacci)
Recall that the Fibonacci sequence (F_n) is defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_n + F_{n+1}$, and that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$.

Let $b \in \mathbb{R}$ with $b \geq 1$ and define a sequence (G_n) by $G_1 = 1$, $G_2 = 1$, and $G_{n+2} = G_n + bG_{n+1}$.

Let $c \in \mathbb{R}$ with $c \geq \phi$. Find b such that $\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = c$.

Solution. Let $c_n = \frac{G_{n+1}}{G_n}$. Then

$$c_{n+1} = \frac{G_{n+2}}{G_{n+1}} = \frac{bG_{n+1} + G_n}{G_{n+1}} = b + \frac{1}{c_n}.$$

Now (c_n) is a Cauchy sequence, so it converges; let $L = \lim c_n$. Since $c_n > 0$ for all n, $L \geq 0$. Then $L = b + \frac{1}{L}$, so

$$L^2 - bL - 1 - 0.$$

Thus

$$L = \frac{b + \sqrt{b^2+4}}{2}.$$

If $L = c$, then $2c = b + \sqrt{b^2+4}$, so $(2c - b)^2 = b^2 + 4$, so $4c^2 - 4bc + b^2 = b^2 + 4$, so

$$\boxed{b = \frac{c^2 - 1}{c}.}$$

□

Problem 2. (Tartaglia)
Recall that Tartaglia viewed the cube x^3 as $(t - u)^3$ to find solutions to cubic equations.

Let $f(x) = x^3 + 3x^2 + 6x - 8$. Find the real zero of f using Tartaglia's cube plus cosa method.

Solution. First we depress the cubic: let $y = x + 1$; then

$$f(x) = f(y-1) = (y-1)^3 + 3(y-1)^2 + 6(y-1) - 8 = y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 + 6y - 6 - 8 = y^3 + 3y - 12.$$

We now solve $y^3 + 3y = 12$. Set $3tu = 3$ and $t^3 - u^3 = 12$, so that $u = \frac{1}{t}$, and $t^3 - \frac{1}{t^3} = 12$. Thus

$$t^6 - 12t^3 - 1 = 0.$$

By the quadratic formula,

$$t^3 = \frac{12 + \sqrt{144+4}}{2} = 6 + \sqrt{37}.$$

Now $u^3 = t^3 - 12 = -6 + \sqrt{37}$. Thus

$$y = t - u = \sqrt[3]{6+\sqrt{37}} + \sqrt[3]{6-\sqrt{37}}.$$

Finally,

$$\boxed{x = \sqrt[3]{6+\sqrt{37}} + \sqrt[3]{6-\sqrt{37}} - 1.}$$

□

Problem 3. (Descartes)
Recall that Descartes used the concept of expanding circles and the ability to compute the number of real solutions to quadratic equations to find tangents.

Find the distance between the curve $x = y^2$ and the point $(3, 0)$ using Descartes' discriminant method.

Solution. A circle of radius r centered at $(3, 0)$ has equation $(x-3)^2 + y^2 = r^2$. The shortest distance to the curve is the radius of a tangential circle, which occurs when then circle intersects the curve in exactly one point.

Intersecting the curve and the circle gives $(x-3)^2+x = r^2$, so $x^2-5x+(9-r^2) = 0$, so $x = \frac{5\pm\sqrt{25-4(9-r^2)}}{2}$. This has exactly one solution when $25 = 4(9-r^2)$, or $r^2 = 9 - \frac{25}{4} = \frac{11}{4}$. Thus the distance is

$$\boxed{r = \frac{\sqrt{11}}{2}.}$$

□

Problem 4. (Napier)
Recall that Napier desired to find a function to convert multiplication into addition. We may use techniques of Calculus unavailable to him to see that he had very little choice. The modern definition is

$$\log x = \int_1^x \frac{dt}{t} \quad \text{and} \quad \log_b(x) = \frac{\log x}{\log b}.$$

Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function which is not constantly zero and satisfies

$$f(ab) = f(a) + f(b) \quad \text{for all } a, b \in (0, \infty).$$

Show that there exists $b \in \mathbb{R}$ such that $f(x) = \log_b(x)$.

Solution. First, note that $f(1) = f(1 \cdot 1) = f(1) + f(1)$; thus $f(1) = 0$.

Fix $t \in (0, \infty)$; we have $f(tx) = f(t) + f(x)$. Differentiating with respect to x gives $tf'(tx) = f'(x)$. In particular, if $x = 1$, we have $tf'(t) = f'(1)$, so $f'(t) = \frac{f'(1)}{t}$. This is true for all $t \in \mathbb{R}$, so

$$\int_1^x f'(t)\,dt = \int_1^x \frac{f'(1)}{t}\,dt.$$

By the Fundamental Theorem of Calculus,

$$f(x) - f(1) = f'(1) \int_1^x \frac{dt}{t} = f'(1) \log x.$$

Since $f(1) = 0$, $f(x) = f'(1) \log x$.

Suppose $f'(1) = 0$; then $tf'(t) = 0$, so $f'(t) = 0$ for all $t \in (0, \infty)$, so f is constant. But $f(1) = 0$, so $f(x) = 0$; this contradicts that f is nonzero. Thus $f'(1) \neq 0$.

Let $b = e^{\frac{1}{f'(1)}}$. Then $f'(1) = \frac{1}{\log b}$, and $f(x) = \frac{\log x}{\log b}$; that is,

$$\boxed{f(x) = \log_b x \quad \text{where } b = e^{\frac{1}{f'(1)}}.}$$

□