HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC V
ARCHIMEDES ON CIRCLES AND SPHERES

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Abstract. Disclaimer: some sections of this document were lifted from the internet, but I no longer remember which ones.

1. Precursors of Archimedes

1.1. **Pythagorean Irrational Numbers.** The Pythagoreans (ca. 500 B.C.) proved the existence of irrational numbers in the form of "incommensurable quantities". This tore at the fabric of their world view, based on the supremacy of whole numbers, and it is legend that the demonstrator of irrational numbers was thrown overboard at sea.

1.2. **Zeno's Paradoxes.** Zeno (ca. 450 B.C.) developed his famous "paradoxes of motion".

1.2.1. *The Dichotomy.* The first paradox asserts the non-existence of motion on the grounds that which is in locomotion must arrive at the half-way stage before it arrives at the goal.

1.2.2. *Achilles and the Tortoise.* The second paradox asserts that it is impossible for Achilles to overtake the tortoise when pursuing it, for he must first reach a point where the tortoise had been, but the tortoise had in the meantime moved forward.

1.2.3. *The Arrow.* The third paradox is that the flying arrow is at rest, which result follows from the assumption that time is composed of moments.

1.2.4. *The Stadium.* The fourth paradox concerns bodies which move alongside bodies in the stadium from opposite directions, from which it follows, according to Zeno, that half the time is equal to its double.

1.3. **Eudoxus Method of Exhaustion.** Eudoxus (ca. 370 B.C.) is remembered for two major mathematical contributions: the *Theory of Proportion*, which filled the gaps in the Pythagorean theories created by the existence of incommensurable quantities, and the *Method of Exhaustion*, which dealt with Zeno's Paradoxes. This method is based on the proposition: *If from any magnitude there be subtracted a part not less than its half, from the remainder another part not less than its half, and so on, there will at length remain a magnitude less than any preassigned magnitude of the same kind.*

Archimedes credits Eudoxus with applying this method to find that the volume of "any cone is on third part of the cylinder which has the same base with the cone and equal height."

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1.4. **Euclid's Elements.** Euclid of Alexandria (ca. 300 B.C.) wrote *The Elements*, which may be the second most published book in history (after the Bible). The work consists of thirteen books, summarizing much of the basic mathematics of the time, spanning plane and solid geometry, number theory, and irrational numbers.

2. Results from Euclid

Result 1. *The circumferences of two circles are to each other as their diameters.*

Using modern notation, this says that if we are given two circles with diameters D_1 and D_2, and circumferences C_1 and C_2, then

$$\frac{C_1}{C_2} = \frac{D_1}{D_2}, \quad \text{whence} \quad \frac{C_1}{D_1} = \frac{C_2}{D_2}.$$

From this, one may conclude that for any given circle, the ratio between the circumference and the diameter is a constant:

$$\frac{C}{D} = p, \quad \text{so} \quad C = pD.$$

We shall call p the *circumference constant*.

Result 2. *The areas of two circles are to each other as the squares of their diameters.*

That is, if A_1 and A_2 represent the area of circles with diameters D_1 and D_2, then

$$\frac{A_1}{A_2} = \frac{D_1^2}{D_2^2}, \quad \text{whence} \quad \frac{A_1}{D_1^2} = \frac{A_2}{D_2^2},$$

which says that there is an *area constant* for any circle:

$$\frac{A}{D^2} = k, \quad \text{so} \quad A = kD^2.$$

However, Euclid doesn't mention, and possibly doesn't realize, that p and k are related.

Result 3. *The volumes of two spheres are to each other as the cubes of their diameters.*

Thus if V_1 and V_2 are the volumes of spheres of diameter D_1 and D_2, then

$$\frac{V_1}{V_2} = \frac{D_1^3}{D_2^3}, \quad \text{whence} \quad \frac{V_1}{D_1^3} = \frac{V_2}{D_2^3};$$

again, one sees that, for again given sphere, there is a *volume constant* m such that

$$\frac{V}{D^3} = m, \quad \text{so} \quad V = mD^3.$$

Note that in each of these three cases (circumference, area, volume), the original statements by Euclid compare like units (e.g. length is to length as area is to area), whereas the modern tendency is to compare aspects of the same object (e.g. area is to length squared).

3. Measurement of a Circle

Proposition 1. *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

Let be C be the circumference, r the radius, and A the area of the circle. Let T be the area of a right triangle with legs of length r and C. Then $T = \frac{1}{2}rC$. Archimedes claims that $A = T$, so $A = \frac{1}{2}rC$.

Lemma 1. *Let h be the apothem and let Q be the perimeter of a regular polygon. Then the area of the polygon is*

$$P = \frac{1}{2}hQ.$$

Proof. Suppose the polygon has n sides, each of length b. Clearly $Q = nb$. Then the area is subdivided into n triangles of base b and height h, so

$$P = n(\frac{1}{2}hb) = \frac{1}{2}hQ.$$

□

Lemma 2. *Consider a circle of area A and let $\epsilon > 0$. Then there exists an inscribed polygon with area P_1 and a circumscribed polygon with area P_2 such that*

$$A - \epsilon < P_1 < A < P_2 < A + \epsilon.$$

Proof. Archimedes simply says: "Inscribe a square, then bisect the arcs, then bisect (if necessary) the halves and so on, until the sides of the inscribed polygon whose angular points are the points of the division subtend segments whose sum is less than the excess of the area of the circle over the triangle." □

Proof of Proposition. By double reductio ad absurdum.

Suppose that $A > T$. Then $A - T > 0$, so there exists an inscribed regular polygon with area P such that $A - P < A - T$. Thus $P > T$. If Q is the perimeter and h the apothem of the polygon, we have

$$P = \frac{1}{2}hQ < \frac{1}{2}rC = T,$$

a contradiction.

On the other hand, suppose that $A < T$. Then $T - A > 0$, so there exists a circumscribed polygon with area P such that $P - A < T - A$. Thus $P < T$. However, if Q is the perimeter and h the apothem of the polygon, we have

$$P = \frac{1}{2}hQ > \frac{1}{2}rC = T,$$

a contradiction.

Therefore, as Archimedes writes, "since then the area of the circle is neither greater nor less than [the area of the triangle], it is equal to it." □

Proposition 2. *The ratio of the circumference of any circle to its diameter is less the* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.

Proof. Inscribe a hexagon. Compute the area:

$$\pi = \frac{C}{D} > \frac{Q}{D} = \frac{6r}{2r} = 3.$$

Archimedes next doubles the number of vertices to obtain a regular dodecagon. The computation of its area requires accurate extraction of $\sqrt{3}$, which Archimedes estimates as

$$\Big(1.732026 \approx \Big)\frac{265}{153} < \sqrt{3} < \frac{1351}{780}\Big(\approx 1.732051\Big),$$

which is impressively close. The Archimedes continues with 24, 48, and finally 96 sides, at each stage extracting more sophisticated square roots.

Next circumscribe a hexagon and continue to 96 sides. □

In decimal notation, my calculator says that

$$3\frac{10}{71} = \frac{223}{71} \approx 3.14085 \quad < \quad \pi \approx 3.14159 \quad < \quad 3\frac{1}{7} = \frac{22}{7} \approx 3.14286.$$

4. On the Sphere and the Cylinder

The two volume work entitled *On the Sphere and the Cylinder* is Archimedes undisputed masterpiece, probably regarded by Archimedes himself as the apex of his career. These two volumes are constructed in a manner similar to Euclid's Elements, in that it proceeds from basic definitions and assumptions, through simpler known results, onto the new discoveries of Archimedes.

Among the results in this work are the following. This first describes the surface area of a sphere in terms of the area of a circle, thus comparing area to area.

Proposition 3. *The surface of any sphere is equal to four times the greatest circle in it.*

Technique of Proof. Double reductio ad absurdum: assumption that the area is more leads to a contradiction, as does assumption that the area is less. One needs to understand the area of a cone to accomplish these estimates (why?). □

Let us translate this into modern notation. Let r be the radius of the sphere and let S be its surface area. Then the radius of the greatest circle in it is πr^2. Thus Archimedes shows that

$$S = 4\pi r^2.$$

The next proposition describes the volume of a sphere in terms of the volume of a cone.

Proposition 4. *Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.*

Note that again, Archimedes has expressed the volume of the sphere in terms of the volume of a known solid; this is because the Greeks did not have modern algebraic notation. Using modern notation, we let r be the radius and let V be the volume of the sphere. The volume of the cone of radius r and height r, as determined by Eudoxus, is $\frac{1}{3}\pi r^3$. Thus

$$V = \frac{4}{3}\pi r^3.$$

In this way, Archimedes found the relationship between the circumference constant p, the area constant k (in *Measurement of a Circle*), and the volume constant m: We have

$$C = pD, \quad A = kD^2, \quad , \text{ and } V = mD^3,$$

and Archimedes has shown (in modern notation) that

$$C = \pi D \qquad \text{(that is, } p = \pi)$$

$$A = \pi r^2 = \pi\Big(\frac{D}{2}\Big)^2 = \frac{\pi}{4}D \qquad (\text{so } k = \frac{\pi}{4})$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\Big(\frac{D}{2}\Big)^3 = \frac{\pi}{6}\pi D^3 \qquad (\text{so } m = \frac{\pi}{6})$$

From here, Archimedes now describes an astounding discovery.

Suppose we have a sphere of radius r, surface area S, and volume V. Inscribe this sphere in a right circular cylinder, whose radius would also be r and whose height would be $2r$. Then the surface area A_{cyl} of the cylinder is simply the areas of the base and top circle, plus the area of the rectangle which forms the tube of the cylinder:

$$A_{\text{cyl}} = 2(\pi r^2) + (2\pi r)(2r) = 6\pi r^2.$$

Thus

$$A_{\text{cyl}} : A_{\text{sph}} = (6\pi r^2) : (4\pi r^2) = 3 : 2.$$

Moreover, the volume of the cylinder is the area of the circular base times the height:

$$V_{\text{cyl}} = (\pi r^2)(2r) = 2\pi r^3.$$

Again, we have

$$V_{\text{cyl}} : V_{\text{sph}} = (2\pi r^3) : (\frac{4}{3}\pi r^3) = 3 : 2.$$

This so intrigued Archimedes that he requested that his tombstone be engraved with a sphere inscribed in a cylinder, together with the ratio 3 : 2. Apparently, Marcellus, the conqueror of Syracuse, was so impressed with Archimedes, that he granted this wish.

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