HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC VIII
THE FIBONACCI SEQUENCE

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Abstract. Sequences play an important role in modern mathematics, and one of the first to investigate them was Leonardo Fibonacci in the twelfth century A.D. We investigate the famous sequence which perpetuates his name.

1. Recursively Defined Sequences

Definition 1. Let X be a set. A *sequence* in X is a function $a : \mathbb{N} \to X$. We normally write a_n to mean $a(n)$, and the entire function is often denoted by $(a_n)_{n=1}^{\infty}$, or simply as (a_n).

Definition 2. Let (a_n) be a sequence in $\mathbb{R}$, and let $L \in \mathbb{R}$. We say that (a_n) *converges* to L, or that L is the *limit* of (a_n), if

$$\text{for every } \epsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |a_n - L| < \epsilon.$$

In this case we write $\lim a_n = L$.

We assume familiarity with the standard properties, and focus on recursively defined sequences. Suppose that we set $a_0 = C$, a fixed constant value, select a function $f : \mathbb{R} \to \mathbb{R}$, and set $a_{n+1} = f(a_n)$ for every n. This uniquely defines a sequence (a_n) of real numbers.

Now it is clear that if we obtain a new sequence (a_{n+1}) from (a_n) by shifting, the limit (should it exist) does not change: $\lim a_{n+1} = \lim a_n$. If (a_n) is a recursively defined sequence such that $a_{n+1} = f(a_n)$ for some *continuous* function f, then $\lim a_{n+1} = f(\lim a_n)$, so if $L = \lim a_n$, we have $L = f(L)$. We use this fact to analyze recursively defined sequences (accept that the following sequences do converge; proving this is typically harder than computing the limit of a recursively defined sequence).

Example 1. Define a sequence (a_n) by $a_0 = 1$ and $a_{n+1} = \frac{a_n}{2}$. Find $\lim a_n$.

Solution. The first few terms of the sequence are $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1/2}{2} = \frac{1}{4}$, $a_3 = \frac{1/4}{2} = \frac{1}{8}$, and so forth; we see that this sequence could have been given as $a_n = \frac{1}{2^n}$. In fact, if $L = \lim a_n$, then $L = \frac{L}{2}$, so $2L = L$, so $L = 0$. □

Example 2. Define a sequence (a_n) by $a_0 = 1$ and $a_{n+1} = \frac{a_n+1}{3}$. Find $\lim a_n$.

Solution. In this case, $a_0 = 1$, $a_1 = \frac{2}{3}$, $a_2 = \frac{5}{9}$, $a_3 = \frac{14}{27}$, $a_4 = \frac{41}{81}$, and so forth. We believe that $a_n = \frac{(3^n+1)/2}{3^n}$; the sequence certainly seems to be approaching $\frac{1}{2}$. In fact, with $L = \lim a_n$, we have $L = \frac{L+1}{3}$, so $3L = L + 1$, so $2L = 1$, and $L = \frac{1}{2}$. □

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Example 3. Define a sequence (a_n) by $a_0 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$. Find $\lim a_n$.

Solution. This sequence formalizes the repeated square root

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}.$$

We have $L = \sqrt{1 + L}$, so $L^2 = 1 + L$, and $L^2 - L - 1 = 0$. Noting the limit must be positive, the quadratic formula gives Ł $= \frac{1+\sqrt{5}}{2}$. That is, L is the golden ratio Φ. The sequence increases to this upper bound. □

Example 4. Define a sequence (a_n) by $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$. Find $\lim a_n$.

Solution. This sequence formalizes the repeated fraction

$$1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{\dots}}}.$$

Let's compute the first few terms of this sequence; we will see an interesting pattern.

- $a_1 = 1$
- $a_2 = 1 + \frac{1}{1} = \frac{1+1}{1} = 2$
- $a_3 = 1 + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}$
- $a_4 = 1 + \frac{2}{3} = \frac{3+2}{3} = \frac{5}{3}$
- $a_5 = 1 + \frac{3}{5} = \frac{5+3}{5} = \frac{8}{5}$
- $a_6 = 1 + \frac{5}{8} = \frac{8+5}{8} = \frac{13}{8}$

We see that, in each case, we add the numerator and denominator and put it over the previous numerator.

We compute that if $L = \lim a_n$, then $L = 1 + \frac{1}{L}$, so $L^2 = L + 1$, so $L^2 - L - 1 = 0$, and $L = \frac{1+\sqrt{5}}{2}$. Actually, the sequence jumps back and forth around Φ, with the even terms less than Φ and the odd terms greater than Φ. □

2. Fibonacci Sequence

Definition 3. Define a sequence (F_n) by setting $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_n + F_{n+1}.$$

Then (F_n) is known as the *Fibonacci sequence*, after the 12^{th} century mathematician Fibonacci, who discovered the sequence while investigating the breeding of rabbits.

The first few terms of the Fibonacci sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 44, 65, 109, 174, 283, 475, \dots$$

Define a sequence (a_n) by $a_0 = 1$ and $a_n = \frac{F_{n+1}}{F_n}$. Then $a_1 = 1$, $a_2 = 2$, $a_3 = \frac{3}{2}$, $a_4 = \frac{5}{3}$; look familiar? Now

$$a_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1} + F_n}{F_{n+1}} = 1 + \frac{1}{a_n};$$

so as we have already seen,

$$\lim \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}.$$

The golden ratio is also involved in the following *generating function* for the Fibonacci sequence:

Proposition 1.

$$F_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big).$$

Solution. The *golden ratio* is the positive solution to the equation $x^2 - x - 1 = 0$; the quadratic formula gives the roots as $\frac{1 \pm \sqrt{5}}{2}$. Set

$$\Phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \Psi = \frac{1-\sqrt{5}}{2}.$$

then Φ and Ψ satisfy the above equation, which produces these identities:

- $\Phi + 1 = \Phi^2$;
- $\Phi - 1 = \frac{1}{\Phi}$;
- $\Psi + 1 = \Phi^2$;
- $\Psi - 1 = \frac{1}{\Psi}$;
- $\Psi = -\frac{1}{\Phi} = 1 - \Phi$;
- $\Phi - \Psi = \sqrt{5}$.

In light of this, what we wish to show can be rewritten as

$$F_n = \frac{1}{\sqrt{5}}\Big(\Phi^n - \Psi^n\Big).$$

We have $F_1 = 1$ and plugging 1 into the above expression produces

$$\frac{1}{\sqrt{5}}\Big(\Phi - \Psi\Big) = \frac{\sqrt{5}}{\sqrt{5}} = 1;$$

therefore the formula is true for $n = 1$.

By strong induction, assume that for $n \geq 3$ we have

$$F_{n-2} = \frac{1}{\sqrt{5}}\Big(\Phi^{n-2} - \Psi^{n-2}\Big);$$
$$F_{n-1} = \frac{1}{\sqrt{5}}\Big(\Phi^{n-1} - \Psi^{n-1}\Big),$$

Then

$$\begin{aligned}
F_n &= F_{n-2} + F_{n-1} \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^{n-2} - \Psi^{n-2}\Big) + \frac{1}{\sqrt{5}}\Big(\Phi^{n-1} - \Psi^{n-1}\Big) \\
&= \frac{1}{\sqrt{5}}\Big((\Phi^{n-2} + \Phi^{n-1}) - (\Psi^{n-2} + \Psi^{n-1})\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^{n-2}(1+\Phi) - \Psi^{n-2}(1+\Psi)\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^{n-2}(\Phi^2) - \Psi^{n-2}(\Psi^2)\Big) \\
&= \frac{1}{\sqrt{5}}\Big(\Phi^n - \Psi^n\Big).
\end{aligned}$$

This completes the proof. □

3. Cauchy Sequences

We now supply a formal proof that the sequence of ratios of the Fibonacci numbers is a Cauchy sequence, and so it does in fact converge.

Definition 4. Let (a_n) be a sequence of real numbers. We say that (a_n) is a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Rightarrow \quad |a_m - a_n| < \epsilon.$$

The proof of the next theorem may be found in books on real analysis.

Theorem 1. (Cauchy Convergence Criterion)
A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proposition 2. *Let (a_n) be a sequence satisfying*

$$|a_{n+1} - a_n| < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Then (a_n) is a Cauchy sequence.

Lemma 1. *Let $m, n \in \mathbb{N}$ with $2 < m < n$. Then*

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{1}{2^m} < \frac{1}{m}.$$

Proof of Lemma. We prove the first inequality by induction on $k = n - m$. If $k = 1$, then our statement reads $\frac{1}{2^{m+1}} < \frac{1}{2^m}$, which is true.

Suppose that our proposition is true for differences of size $k - 1$. Then

$$\sum_{i=m+2}^{n} \frac{1}{2^i} < \frac{1}{2^{m+1}}.$$

Adding $\frac{1}{2^{m+1}}$ to both sides gives

$$\sum_{i=m+1}^{n} \frac{1}{2^i} < \frac{2}{2^{m+1}} = \frac{1}{2^m}.$$

For the second inequality, it suffices to show that for $m > 2$ we have $m < 2^m$. For $m = 3$, we have $3 < 4$. By induction, $m - 1 < 2^{m-1}$. Then $m < 2^{m-1} + 1 < 2^{m-1} + 2^{m-1} = 2^m$. □

Proof of Proposition. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\frac{1}{\epsilon} < N$. Let $m, n > N$; assume that $n > m$. Then

$$\begin{aligned}
|a_n - a_m| &= |a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{m+1} - a_m| \\
&\leq |a_n - a_{n-1}| + \cdots + |a_{m+1} - a_m| \\
&< \frac{1}{2^{n-1}} + \cdots + \frac{1}{2^m} \\
&< \frac{1}{2^{m-1}} \\
&< \frac{1}{m-1} \leq \frac{1}{N} < \epsilon.
\end{aligned}$$

This shows that (a_n) is a Cauchy sequence. □

Proposition 3. *Define a sequence* (a_n) *by*

$$a_n = \frac{F_{n+1}}{F_n}.$$

Then (a_n) *is a Cauchy sequence which converges to* $\frac{1+\sqrt{5}}{2}$*.*

Proof. To show that (a_n) is a Cauchy sequence, it suffices to show that

$$|a_{n+1} - a_n| < \frac{1}{2^{n-1}}.$$

To do this, we first show that $F_n F_{n+1} > 2^{n-1}$ for $n \geq 3$. For $n = 3$, we have $F_3 F_4 = 2 \cdot 3 > 4$. By induction, assume that $F_{n-1} F_n > 2^{n-2}$. Clearly (F_n) is a nondecreasing sequence, so

$$F_n F_{n+1} = F_n^2 + F_n F_{n-1} \geq 2 F_n F_{n-1} > 2^{n-1}.$$

Next we show that $|F_n F_{n+2} - F_{n+1}^2| = 1$ for $n \geq 1$. For $n = 1$, we have $|F_1 F_3 - F_2^2| = 2 - 1 = 1$. By induction, assume that $|F_{n-1} F_{n+1} - F_n^2| = 1$. Then

$$\begin{aligned} |F_n F_{n+2} - F_{n+1}^2| &= |F_n(F_n + F_{n+1}) - F_{n+1}^2| \\ &= |F_n^2 + F_n F_{n+1} - F_{n+1}^2| \\ &= |F_n^2 - F_{n+1}(F_{n+1} - F_n)| \\ &= |F_n^2 - F_{n+1} F_{n-1}| \\ &= 1. \end{aligned}$$

Now

$$\begin{aligned} |a_{n+1} - a_n| &= \left| \frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n} \right| \\ &= \left| \frac{F_{n+2} F_n - F_{n+1}^2}{F_n F_{n+1}} \right| \\ &= \left| \frac{1}{F_n F_{n+1}} \right| \\ &< \frac{1}{2^{n-1}}. \end{aligned}$$

Since (a_n) is a Cauchy sequence, it converges; let $L = \lim(a_n)$. Since a_n is positive for all n, $L \geq 0$. Now

$$a_{n+1} = \frac{F_{n+2}}{F_{n+1}} = \frac{F_n + F_{n+1}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{a_n}.$$

Taking the limit of both sides of this equation, we have $L = 1 + \frac{1}{L}$. Thus

$$L^2 - L - 1 = 0.$$

The positive solution to this quadratic equation is

$$L = \frac{1 + \sqrt{5}}{2}.$$

□

Proposition 4. *Let $b \in \mathbb{R}$, $b \geq 1$, and define a sequence (G_n) by $G_1 = 1$, $G_2 = 1$, and $G_{n+2} = G_n + bG_{n+1}$. Define a sequence (c_n) by*

$$c_n = \frac{G_{n+1}}{G_n}.$$

Then (c_n) is a Cauchy sequence.

Proof. To show that (c_n) is a Cauchy sequence, it suffices to show that

$$|c_{n+1} - c_n| < \frac{b}{2^{n-1}}.$$

To do this, we first show that $G_n G_{n+1} > 2^{n-1}$ for $n \geq 3$. For $n = 3$, we have $G_3 G_4 = (b+1)(b^2+b+1) > 4$. By induction, assume that $G_{n-1} G_n > 2^{n-2}$. Clearly (G_n) is a nondecreasing sequence, so

$$G_n G_{n+1} = bG_n^2 + G_n G_{n-1} \geq G_n^2 + G_n G_{n-1} \geq 2G_n G_{n-1} > 2^{n-1}.$$

Next we show that $|G_n G_{n+2} - G_{n+1}^2| = b$ for $n \geq 1$. For $n = 1$, we have $|G_1 G_3 - G_2^2| = b + 1 - 1 = b$. By induction, assume that $|G_{n-1} G_{n+1} - G_n^2| = b$. Then

$$\begin{aligned}
|G_n G_{n+2} - G_{n+1}^2| &= |G_n(G_n + bG_{n+1}) - G_{n+1}^2| \\
&= |G_n^2 + bG_n G_{n+1} - G_{n+1}^2| \\
&= |G_n^2 - G_{n+1}(G_{n+1} - bG_n)| \\
&= |G_n^2 - G_{n+1} G_{n-1}| \\
&= b.
\end{aligned}$$

Now

$$\begin{aligned}
|c_{n+1} - c_n| &= \Big|\frac{G_{n+2}}{G_{n+1}} - \frac{G_{n+1}}{G_n}\Big| \\
&= \Big|\frac{G_{n+2} G_n - G_{n+1}^2}{G_n G_{n+1}}\Big| \\
&= \Big|\frac{b}{G_n G_{n+1}}\Big| \\
&< \frac{b}{2^{n-1}}.
\end{aligned}$$

Thus (c_n) is a Cauchy sequence. □

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