HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC XII
NEWTON AND POWER SERIES

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Historical Background

Bonaventura Cavalieri (Italian 1598-1647)

- Introduced logarithms into Italy
- Wrote books on mathematics, optics, and astronomy
- Wrote *Geometria indivisibilibus*, published 1646, devoted to the *method of indivisibles* (parallel line segments in a plane region, of parallel region constituting a volume) producing *Cavalieri's principles.*

Cavalieri's principles are:

(a) If two planar pices are included between a pair of parallel lines, and if the lengths of the two segments cut by them on any line parallel to the including lines are always in a given ratio, then the areas of the two planar pieces are also in this ratio.

(b) If two solids are included between a pair of parallel palnes, and if the areas of the two sections cut by them on any plane parallel to the including planes are always in a given ratio, then the volumes of the two solids are also in this ratio.

John Wallis (English 1619-1703)

- Conics as degree two equations
- Extended methods of Descartes and Cavalieri
- Introduced symbol ∞
- Computed $\int_0^1 \sqrt{1-x^2}\,dx$ as an infinite product
- Computed arc lengths

Isaac Barrow (English 1630 - 1677)

- Proposed the "differential triangle", which is similar to Fermat's method but emphasizing slope.
- States and proved a version of the Fundamental Theorem of Calculus in *Lectiones optical et geometrical* (1670)

Date: December 10, 2008.

Isaac Newton (English 1642-1727)

- (1665) Early discoveries at Cambridge (1665-1666)
  - Generalized binomial theorem
  - Method of Fluxions (differential calculus)
  - Optics
  - Gravitation
- (1665) Compares decimal numbers to power series of variables with the analogy `series:algebra::decimal:arithmetic`
- (1668) Quadrature of the Hyperbola, published by Mercator in 1668
$$\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} (= \log(1+t))$$
- (1669) Barrow resigned Lucasion chair, which was taken over by Newton. As such, Newton had to give lectures: "If he had an audience, they lasted 30 minutes; otherwise he spoke to the walls for 15 minutes and then left."
- (1671) Method of Fluxions written, but not published until 1736
- (1675) Corpuscular Theory of Light [Huygen's developed the wave theory]
- (1679) Verified gravitational equation $F \sim \frac{m_1 m_2}{r^2}$, and derived Kepler's laws of planetary motion from this.
- (1684) Halley convinced him to write ... wrote first book of *Philosohiae naturalis principia mathematica*
- (1687) Published at Halley's expense

Gottfried Leibniz (German 1646-1716)

- Diplomat, philosopher, lawyer
- Developed Calculus separately from Newton
- Notation: $\frac{dy}{dx}$, $\int y\,dx$ ($\int$ is an "S" from Latin word "summa", as a sum of Cavalieri's indivisibles)
- Introduced the word "function" and was the first to think in function terms.
- Distinction between algebraic and transcendental functions
- Preferred "closed form" to infinite series
- Also developed the theory of determinants

The views of Newton and Leibnitz differ on integration:

$$\int f(x)\,dx \quad \text{means} \quad \begin{cases} \text{Leibniz: find the closed form antiderivative} \\ \text{Newton: express } f(x) \text{ as a power series and lift each term} \end{cases}$$

1. Sequences

A *sequence* of real numbers is a function

$$a : \mathbb{N} \to \mathbb{R};$$

if $n \in \mathbb{N}$, we typically write a_n instead of $a(n)$. We denote the sequence $a : \mathbb{N} \to \mathbb{R}$ by (a_n).

Let (a_n) be a sequence and let $L \in \mathbb{R}$. We say that (a_n) *converges to* L if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$N < n \Rightarrow |a_n - L| < \epsilon.$$

If a sequence converges to a real number L, we say it is *convergent*, and we say that L is the *limit* of the sequence; we may write

$$L = \lim_{n\to\infty} a_n.$$

It is a fact that limits, when they exist, are unique.

If a sequence does not converge to a real number L, it is *divergent*.

One may form sums and products of sequences:

$$(a_n) + (b_n) = (a_n + b_n)$$
$$(a_n)(b_n) = (a_n b_n)$$

If (a_n) converges to L_1 and b_n converges to L_2, then $(a_n)+(b_n)$ converges to L_1+L_2 and $(a_n)(b_n)$ converges to $L_1 L_2$.

If (a_n) is nonzero and converges to L, then

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}.$$

Let (a_n) be a sequence. We say (a_n) is *increasing* if $a_m \leq a_n$ whenever $m \leq n$; we say that (a_n) is *decreasing* if $a_m \geq a_n$ whenever $m \leq n$; we say that (a_n) is *monotone* if it is either increasing or decreasing. We say that (a_n) is *bounded* if there exists a positive real number B such that $a_n \in [-B, B]$ for all $n \in \mathbb{N}$.

Fact 1. (Bounded Monotone Convergence Rule)
A bounded monotone sequence of real numbers converges.

Fact 2. (Squeeze Law)
If (a_n) and (b_n) both converge to L, and $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$, then (c_n) converges to L.

2. Series

Let (a_n) be a sequence. Then n^{th} *partial sum* of this series is

$$s_n = \sum_{i=0}^{n}.$$

A *series* is a sequence of the form (s_n), where s_n is the n^{th} partial sum of some sequence (a_n). Such a series may be denoted by $\sum a_n$.

A series $\sum a_n$ converges if the sequence of partial sums converges. In this case, we let $\sum_{n=0}^{\infty} a_n$ denote the limit of the series.

We say a series $\sum a_n$ *converges absolutely* if the associated series $\sum |a_n|$ converges. If a series converges absolutely, then it converges.

One may form sums and products of series:

$$\sum a_n + \sum b_n = \sum (a_n + b_n);$$
$$\sum a_n \sum b_n = \sum_{n=1}^{\infty} \Big(\sum_{j=1}^{n} a_j b_{n-j} \Big).$$

If $\sum a_n$ converges to S_1 and $\sum b_n$ converges to S_2, then $\sum a_n + \sum b_n$ converges to $S_1 + S_2$ and $\sum a_n \sum b_n$ converges to $S_1 S_2$.

Fact 3. (Limit Test)
If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Reason. Set $s = \sum a_n$. Note that $a_n - s_n - s_{n-1}$, where $s_n = \sum_{i=1}^{n}$, so that $s = \lim s_n$. Now (s_{n-1}) is a sequence, whose limit is also clearly s. Thus

$$\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = s - s = 0.$$

$\square$

Fact 4. (Comparison Test)
Let $\sum c_n$ be a convergent series and let $\sum d_n$ be a divergent series.

- **(a)** If $0 \leq a_n \leq c_n$ for all $n \in \mathbb{N}$, then $\sum a_n$ converges.
- **(b)** If $0 \leq d_n \leq b_n$ for all $n \in \mathbb{N}$, then $\sum b_n$ diverges.

Fact 5. (Geometric Series Test)
Let $r \geq 0$.

- **(a)** If $r < 1$, then $\sum r^n$ converges to $\frac{1}{1-r}$.
- **(b)** If $r \geq 1$, then $\sum r^n$ diverges.

Reason. Note that $1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$; therefore $\frac{1-x^n}{1-x} = \sum_{i=0}^{n-1} x^i$. If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$; thus

$$\begin{aligned}
\sum_{i=0}^{\infty} x^n &= \lim_{n\to\infty} \sum_{i=0}^{n-1} x^i \\
&= \lim_{n\to\infty} \frac{1-x^n}{1-x} \\
&= \frac{1}{1-x}.
\end{aligned}$$

$\square$

Fact 6. (Alternating Series Test)
Let (a_n) be a decreasing sequence of nonnegative real numbers which converges to zero. Then $\sum(-1)^n a_n$ converges.

Reason. Note that $0 \leq s_2 \leq s_4 \leq s_6 \leq \cdots \leq a_1$. Thus (s_{2n}) is a bounded monotone sequence, and so it converges, say to s. Then $\lim s_{2n+1} = \lim s_{2n} + \lim a_{2n+1} = s + 0 = s$. □

Fact 7. (Ratio Test)
Let (a_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then $\sum a_n$ converges if $L < 1$ and $\sum a_n$ diverges if $L > 1$.

Reason. Suppose $0 < L < 1$. Select r such that $0 < L < r < 1$. Let N be so large that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \text{for} \quad n \geq N.$$

Then $|a_{n+1}| < r|a_n|$, for $n \geq N$.

In particular, $|a_{N+1}| < r|a_N|$, $|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$, and in general, $|a_{N+k}| < r^k|a_N|$. Now

$$\sum_{k=1}^{\infty} |a_n| < \sum_{k=1}^{\infty} |a_N| r^k,$$

which converges. □

Fact 8. (Root Test)
Let (a_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \sqrt{n} a_n = L.$$

Then $\sum a_n$ converges if $L < 1$ and $\sum a_n$ diverges if $L > 1$.

3. Power Series

A *power series* centered at $x_0 \in A$, where $A \subset \mathbb{R}$, is a function

$$f : A \to \mathbb{R}$$

which can be expressed in the form

$$f(x) = \sum_{i=0}^{\infty} a_n (x - x_0)^n.$$

Here, A is the set of points $x \in A$ where $f(x)$ converges. First we want to understand the set A. If we say $R \in [0, \infty]$, we mean that R is either a nonnegative real number or $R = \infty$.

Fact 9. Let $f(x) = \sum a_n (x - x_0)^n$ be a power series. Then there exists a number $R \in [0, \infty]$ such that

(a) $f(x)$ converges absolutely if $|x - x_0| < R$;
(b) $f(x)$ diverges if $|x - x_0| > R$.

This number R is called the *radius of convergence* of f.

We may compute the radius of convergence using our knowledge of series; in particular, the ratio test is useful. Suppose that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

Let $r = x - x_0$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}(x - x_0)^{n+1}|}{|a_n (x - x_0)^n|} = \lim_{n\to\infty} |\frac{a_{n+1} r}{a_n}| = rL.$$

Now $f(x) = \sum a_n (x - x_0)^n$ converges at the point x if $rL < 1$, which happens if $r < \frac{1}{L}$. On the other hand, if $r > \frac{1}{L}$, then $f(x)$ diverges. Thus the radius of convergence is $R = \frac{1}{L}$, i.e.,

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

Similarly, we can use the root test to derive the formula

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}}.$$

Let $f(x) = \sum a_n (x - x_0)^n$ be a power series and let R be its radius of convergence. The *interval of convergence* of $f(x)$ the open interval $I = (x_0 - R, x_0 + R)$; if $R = \infty$, we take this to mean $I = \mathbb{R}$.

4. Power Series Algebra

We have defined power series as functions, and they behave very much like polynomial functions in a couple of ways.

Two functions are equal if and only if they have the same domain and range and take on the same value at every point in the domain. The following gives a useful condition for two power series to be equal; this condition is directly analogous to the condition for polynomial functions.

Fact 10. Let $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n$ be two power series centered at x_0. Then $f = g$ as functions if and only if $a_n = b_n$ for every $n \in \mathbb{N}$.

The sum and product of functions is defined pointwise: $(f + g)(x) = f(x) + g(x)$, and $(fg)(x) = f(x)g(x)$. In the polynomial case, these can be obtained by distribution and reassociation. This remains true for power series.

Fact 11. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ be two power series centered at x_0. Then $f + g$ and fg are power series given by

$$(f + g)(x) = \sum_{n=0}^{\infty}(a_n + b_n)(x - x_0)^n$$

and

$$(fg)(x) = \sum_{n=0}^{\infty} \Big[\sum_{i=0}^{n} a_i b_{n-i}\Big](x - x_0)^n.$$

5. Shifting the Index of a Power Series

Let $k \in \mathbb{Z}$ and consider the infinite sum

$$\sum_{n=k}^{\infty} a_n(x - x_0)^n.$$

If $k < 0$, then this is not a power series. However, if $k > 0$, we consider this to be the power series by understanding that $a_i = 0$ for $i = 0, \dots, k - 1$.

It is sometimes convenient to shift the index of a power series. The following is a formula for doing so:

$$\sum_{n=k}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_{n-k}(x - x_0)^{n-k}.$$

6. Differentiation of Power Series

It seems reasonable one may pass the differentiation operator inside the infinite sum:

$$\begin{aligned}\frac{d}{dx}\sum_{n=0}^{\infty} a_n(x-x_0)^n &= \sum_{n=0}^{\infty}\frac{d}{dx}(a_n(x-x_0)^n)\\ &= \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}.\end{aligned}$$

This is indeed the case.

Fact 12. Let $f(x) = \sum a_n(x-x_0)^n$ be a power series. Then f is differentiable in its radius of convergence, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}.$$

Let $f(x) = \sum a_n(x-x_0)^n$ be a power series. We know attempt to find a formula which relates the derivatives of f to the coefficients a_n.

Note that for any power series, if we evaluate it at its center, we pick out the first coefficient because all of the other terms vanish at the center. By successively differentiating the power series, we shift the coefficients to the left. At each stage we write the first few terms to see how this goes.

Start with

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots;$$

thus $f(x_0) = a_0$, since all of the other terms in the series are of the form $a_n(x-x_0)^n$ and so they vanish at x_0.

Now $f'(x)$ is the power series

$$f'(x) = a_1 + \frac{a_2}{2}(x-x_0) + \frac{a_3}{3}(x-x_0)^2 + \frac{a_4}{4}(x-x_0)^3 + \dots;$$

by plugging in x_0, we pick off the constant coefficient; this time, we get $f'(x_0) = a_1$.

Differentiating again shifts the coefficients to the left to get

$$f''(x) = \frac{a_2}{2} + \frac{a_3}{2\cdot 3}(x-x_0) + \frac{a_4}{3\cdot 4}(x-x_0)^2 + \frac{a_5}{4\cdot 5}(x-x_0)^3 + \dots;$$

thus $f''(x_0) = \frac{a_2}{2}$.

One more time gives

$$f'''(x) = \frac{a_3}{2\cdot 3} + \frac{a_4}{2\cdot 3\cdot 4}(x-x_0) + \frac{a_5}{3\cdot 4\cdot 5}(x-x_0)^2 + \frac{a_6}{4\cdot 5\cdot 6}(x-x_0)^3 + \dots;$$

thus $f'''(x) = \frac{a_3}{6}$.

We now see the pattern; by the n^{th} differentiation, the n^{th} coefficient has moved into the constant coefficient position, but has been divided by $n!$ along the way. This gives us our main formula regarding power series.

Fact 13. Let $f(x) = \sum a_n(x-x_0)^n$ be a power series with positive radius of convergence. Then

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

7. Taylor Series and Analytic Functions

Let $I \subset \mathbb{R}$ be an open interval and let $g : I \to \mathbb{R}$ be a smooth function on I. Let $x_0 \in I$.

The *Taylor series* of f expanded around x_0 There is a natural power series associated to the function g and the point x_0, called the *Taylor series* of f expanded around x_0, and defined by

$$f(x) = \sum a_n(x - x_0)^n,$$

where

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

Note that if g is already a power series, it is equal to the associated power series around any point $x_0 \in I$.

We say that g is *analytic at* x_0 if there exists a sequence (a_n) of real numbers and a real number $R > 0$ such that for every $x \in I \cap (x_0 - R, x_0 + R)$, the power series

$$f(x) = \sum a_n(x - x_0)^n$$

converges, and $f(x) = g(x)$. We say that g is *analytic* if it is analytic at every point in I.

We see that g is analytic when it is equal to its Taylor series expansion around any point, and that the constant R above can be taken to be the radius of convergence of the Taylor series.

Fact 14. Let $f : I \to \mathbb{R}$ be analytic at $x_0 \in I$ with radius of convergence R. Let $x_1 \in I \cap (x_0 - R, x_0 + R)$. Then f is analytic at x_1, with radius of convergence greater than or equal to $\min\{x_1 - x_0 + R, x_0 + R - x_1\}$.

Let $f : I \to \mathbb{R}$ and $g : I \to R$ be analytic, and let $c \in \mathbb{R}$ be a constant. Then $f + g : I \to \mathbb{R}$, $cf : I \to \mathbb{R}$, and $fg : I \to \mathbb{R}$ are also analytic. Quotients of analytic functions are analytic in their domain of definition (with one caveat we will see later). If f and g are expanded around the same point $x_0 \in I$, the radius of convergence of these derived functions is at least as large as the minimum radius of convergence between f and g.

Let $\mathcal{A}(I) = \{f : I \to \mathbb{R} \mid f \text{ is analytic }\}$. Then $\mathcal{A}(I)$ is a vector space over $\mathbb{R}$.

Let $f(x) = \sum a_n(x - x_0)^n$ be analytic. We say that $f(x)$ is *entire* if the radius of convergence of f around x_0 is infinite. When this is the case, the radius of convergence of f expanded around any real number is still infinite.

The following functions are entire: constants, polynomials, exp, sin, and cos.

8. Standard Examples

Example 1. Find the Taylor expansion for $f(x) = \exp(x)$ around 0 and its radius of convergence.

Solution. All derivatives of f are the same. Thus the coefficients are simply

$$a_n = \frac{f^n(0)}{n!} = \frac{\exp(0)}{n!} = \frac{1}{n!}.$$

Thus

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The radius of convergence is

$$R = \lim_{n\to\infty} \frac{1/n!}{1/(n+1)!} = \lim_{n\to\infty} n+1 = \infty.$$

Thus exp is entire. □

Example 2. The Taylor expansion of $\sin(x)$ around 0 is given by

$$\sin(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Thus sin is entire by the alternating series test.

Example 3. The Taylor expansion of $\cos(x)$ around 0 is given by

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

Thus cos is entire by the alternating series test.

Example 4. Find the Taylor expansion for $f(x) = \tan x$ around 0 and its radius of convergence.

Solution. First we take derivatives:

$$f'(x) = \sec^2 x; \quad f''(x) = 2\sec^2 x \tan x; \quad f'''(x) = 4\sec^2 x \tan^2 x + \sec^4 x.$$

Now we evaluate at 0:

$$f(0) = 0; \quad f'(0) = 1; \quad f''(0) = 0; \quad f'''(0) = 1.$$

□

Example 5. The Taylor expansion of $\log(1+x)$ around 0 is given by

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

Its radius of convergence is 1.

Example 6. Compute the Taylor expansion of $f(x) = \frac{1}{1+x}$ around $x_0 = 0$ and find its radius of convergence.

Solution. First, we differentiate until we begin to see a pattern. Then we plug in 0.

$$\begin{aligned}
f(x) &= \frac{1}{1+x} & f(0) &= 1 = 0! \\
f'(x) &= \frac{-1}{(1+x)^2} & f'(0) &= -1 = -1! \\
f''(x) &= \frac{2}{(1+x)^3} & f''(0) &= 2 = 2! \\
f'''(x) &= \frac{-6}{(1+x)^4} & f'''(0) &= -6 = -3! \\
f^{iv}(x) &= \frac{24}{(1+x)^5} & f^{iv}(0) &= 24 = 4!
\end{aligned}$$

We see that $f^{(n)}(0) = (-1)^n n!$. Then $a_n = (-1)^n$, and

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n.$$

There is an easier way to do this by using the geometric series. Let $r = -x$; then

$$f(x) = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

The radius of convergence is 1. □

We see that, in this example, the radius of convergence centered at x_0 is the distance from x_0 to the nearest point of discontinuity.

Example 7. Compute the Taylor expansion of $g(x) = \frac{1}{1+x^2}$ around $x_0 = 0$ and find its radius of convergence.

Solution. Note that $g(x) = f(x^2)$, where $f(x) = \frac{1}{1+x}$. Then

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

This is a power series with the coefficients of the odd terms all equal to zero. Its radius of convergence is still equal to 1. □

In this example, the function $g(x)$ is continuous and analytic at every point $x \in \mathbb{R}$. Then why does it have a finite radius of convergence? We answer this after one more example.

Example 8. Compute the Taylor expansion of $f(x) = \arctan(x)$ around $x_0 = 0$ and find its radius of convergence.

Solution. Let $f(x) = \arctan(x)$. Then $f'(x) = \frac{1}{1+x^2}$; view this as a geometric series. This produces

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} \\ &= \frac{1}{1-(-x^2)} \\ &= \sum_{n=0}^{\infty}(-x^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} \\ &= 1 - x^2 + x^4 - x^6 + x^8 + \cdots . \end{aligned}$$

Now

$$\begin{aligned} f(x) &= \int \frac{1}{1+x^2}\,dx \\ &= \int \Big(\sum_{n=0}^{\infty}(-1)^n x^{2n} \Big)\,dx \\ &= \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx \\ &= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \end{aligned}$$

□

9. Binomial Theorem

Power series are a generalization of polynomials. The extent to which this is true is illuminated by the generalized binomial theorem, discovered by Isaac Newton in the seventeenth century.

Let $n \in \mathbb{N}$, and define $\binom{n}{i}$ to be the number of possible ways to choose a set of i things from a set of n things. We see that there are n choices for the first thing, $n-1$ choices for the second, and so forth, until finally there are $n-i+1$ choices for the i^{th} thing. Thus there are $n(n-1)\cdots(n-i) = \frac{n!}{(n-i)!}$ possibilities for choosing i things, *in a certain order*. There are $i!$ possible different orders for the same set of i things, so altogether we have

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

These numbers are exactly those which are produces via Pascal's Triangle, and are called the *binomial coefficients*. This name comes from the binomial theorem for positive integers, which we state as

$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

This may be thought of as follows: multiplying $x+1$ by itself n times using distribution involves 2^n multiplications of n things, either x or 1 from each of the n copies of the $(x+1)$'s that are being multiplied. Each such multiplication involves choosing either x or 1 from each binomial $(x+1)$. There are $\binom{n}{i}$ different ways of selecting i x's and $(n-i)$ 1's. When we collect like terms, the coefficient of x^i is the number of x^i's occurring in the sum; this is $\binom{n}{i}$.

Suppose that $i > n$; there are zero ways of choosing a set of i items from of a set of n items, so the natural definition in this case is $\binom{n}{i} = 0$. In this case, we may write $(x+1)^n$ as a power series:

$$(x+1)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

because $\binom{n}{i} = 0$ for $i > n$.

Newton saw this, and generalized it in the following fashion, which we explain in modern notation.

Let $\alpha \in \mathbb{R}$, and consider the function $f(x) = (x+1)^\alpha$, so that

$$\begin{aligned} f(x) &= (x+1)^\alpha, \\ f'(x) &= \alpha(x+1)^{\alpha-1}, \\ f''(x) &= \alpha(\alpha-1)(x+1)^{\alpha-2}, \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(x+1)^{\alpha-3}, \end{aligned}$$

and so forth. Generalizing the binomial coefficients, for $i \in \mathbb{N}$ define

$$\binom{\alpha}{i} = \frac{\prod_{j=0}^{i-1}(\alpha - j)}{i!}.$$

Here we use the convention that the empty product is 1, so $\binom{\alpha}{0} = 1$. Then we see that

$$f^{(i)}(x) = i!\binom{\alpha}{i-1}(x+1)^{\alpha-i}.$$

Evaluating this at $x_0 = 0$, we have

$$f^{(i)}(0) = i!\binom{\alpha}{i-1}.$$

Thus the i^{th} coefficient of the Taylor expansion of f around $x_0 = 0$ is

$$a_i = \frac{f^{(i)}(0)}{i!} = \binom{\alpha}{i},$$

so the Taylor series of $f(x)$ is

$$f(x) = \sum_{i=0}^{\infty}\binom{\alpha}{i}x^i;$$

this is known as the *binomial series*. The radius of convergence of the binomial series is

$$R = \lim_{i\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-i+1)/i!}{\alpha(\alpha-1)\cdots(\alpha-i)/(i+1)!}\right| = \lim_{i\to\infty}\left|\frac{i+1}{\alpha-i}\right| = \lim_{i\to\infty}\left|\frac{1+\frac{1}{i}}{1-\frac{\alpha}{i}}\right| = 1.$$

For example, let $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$. Then

$$\begin{aligned}
\sqrt{1-x} &= \sum_{i=0}^{\infty}\binom{1/2}{i}(-x)^i \\
&= 1 + \frac{1}{2}(-x) + \Big(\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)\frac{(-x)^2}{2!} + \Big(\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\frac{(-x)^3}{3!} \\
&+ \Big(\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\Big(-\frac{5}{2}\Big)\frac{(-x)^4}{4!} \\
&+ \Big(\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\Big(-\frac{5}{2}\Big)\Big(-\frac{7}{2}\Big)\frac{(-x)^5}{5!} + \cdots \\
&= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 + \cdots
\end{aligned}$$

Newton used this to estimate $\sqrt{3}$, using that

$$\sqrt{3} = \sqrt{4-1} = 2\sqrt{1-\frac{1}{4}}.$$

Letting $x = \frac{1}{4}$, we have

$$\begin{aligned}
\sqrt{3} &\approx 2\Big(1 - \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{8}\cdot\frac{1}{16} - \frac{1}{16}\cdot\frac{1}{64} - \frac{5}{128}\cdot\frac{1}{256} - \frac{7}{256}\cdot\frac{1}{1024}\Big) \\
&= 2 - \frac{1}{4} - \frac{1}{64} - \frac{1}{512} - \frac{5}{16384} - \frac{7}{131072} \\
&\approx 1.732063293.
\end{aligned}$$

Let $s = 1.732063293$; to nine decimal places, the actual value is $\sqrt{3} \approx 1.732050808$. To get more accuracy, Newton could have just used a few more terms.

10. Newton's Approximation for π

Consider the function $f(x) = \sqrt{x - x^2}$. The graph of this function is the upper half of a circle of radius one half centered at the point $(\frac{1}{2}, 0)$. Compute the area under the curve between $x = 0$ and $x = \frac{1}{4}$ in two ways; using the method of "fluxions", and then using geometry.

10.1. **Area by Fluxions.** The method of fluxions, expressed in modern language, consists of expanding functions into their Taylor series, the differentiating or integrating term by term. The area of which speak is

$$\begin{aligned}
\int_0^{\frac{1}{4}} f(x)\,dx &= \int \sqrt{x}\sqrt{1-x}\,dx\Big|_{\frac{1}{4}} \\
&\approx \int \sqrt{x}\Big(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5\Big)\,dx\Big|_{\frac{1}{4}} \\
&= \int \Big(x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \frac{7}{256}x^{11/2}\Big)\,dx\Big|_{\frac{1}{4}} \\
&= \frac{2}{3}x^{3/2} - \frac{1}{5}x^{5/2} - \frac{1}{28}x^{7/2} - \frac{1}{72}x^{9/2} - \frac{5}{704}x^{11/2} - \frac{7}{1664}x^{13/2}\Big|_{\frac{1}{4}} \\
&= \frac{2}{3}\Big(\frac{1}{2}\Big)^3 - \frac{1}{5}\Big(\frac{1}{2}\Big)^5 - \frac{1}{28}\Big(\frac{1}{2}\Big)^7 - \frac{1}{72}\Big(\frac{1}{2}\Big)^9 - \frac{5}{704}\Big(\frac{1}{2}\Big)^{11} - \frac{7}{1664}\Big(\frac{1}{2}\Big)^{13} \\
&= \frac{1}{12} - \frac{1}{1670} - \frac{1}{3584} - \frac{1}{36864} - \frac{5}{1441792} - \frac{7}{13631488} \\
&\approx 0.076773207.
\end{aligned}$$

Let $a = 0.076773207$; this is our approximation for the area of the region being considered.

10.2. **Area by Geometry.** Let $O = (0, 0)$, $A = (\frac{1}{4}, 0)$, $B = (\frac{1}{2}, 0)$, and $C = (\frac{1}{4}, \frac{\sqrt{3}}{4})$. Then C is on the semicircle $y = \sqrt{x - x^2}$. The sector OBC is one sixth of this circle if radius $\frac{1}{2}$, so its area is $\frac{\pi}{24}$. The triangle ABC has area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{4}$; thus the area of the sector is $\frac{\pi}{24} - \frac{\sqrt{3}}{32}$. This is approximately equal to a, so

$$\pi \approx 24(a + \frac{\sqrt{3}}{32}) \approx 24(a + \frac{s}{32}),$$

where $s = 1.732063293$ is the approximation for $\sqrt{3}$ we obtained in the last section. Thus

$$\pi \approx 3.141604438.$$

Newton actually carried this approximation out using nine terms of the binomial expansion, and obtained an estimate for π which was accurate to the seventh decimal place.

11. Analytic Functions and Complex Numbers

Why do some functions have a finite radius of convergence? For example, we know that $\tan x$ is not defined wherever $\cos x = 0$, for example at $x = \frac{pi}{2}$, so if we expand $\tan x$ around $x_0 = 0$, we are bound to see that the radius of convergence is no bigger than $\frac{\pi}{2}$; on the other hand, since $\sin x$ and $\cos x$ are entire and $\cos x$ is nonzero in $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, we expect that $\tan x$ is analytic in I so the radius of convergence of the expansion around 0 should be exactly $\frac{\pi}{2}$, which turns out to be the case.

However, this doesn't explain the radius of convergence of the function $f(x) = \frac{x}{1+x^2}$, which is analytic in the interval $I = (-1, 1)$, but when expanded around zero has a radius of convergence of only 1. The numerator and denominator are analytic and the denominator is nonvanishing for all real numbers x; why isn't f analytic? To understand this, we must expand our vision to the complex plane.

Our entire theory of sequences, series, power series, and Taylor series generalizes to use of complex numbers. A complex power series has a *disk of convergence*; if

$$f(z) = \sum a_n (z - z_0)^n,$$

where $a_n, z_0 \in \mathbb{C}$, then f converges in a disk around z_0 of radius R, where R is the radius of convergence as computed above (the absolute value of a complex number is its modulus).

The answer to our question is: the radius of convergence is the distance to the nearest *nonremovable complex singularity.* Let us examine what this means.

12. Laurent Series

Let $I \subset \mathbb{R}$ be an open interval. Let $x_0 \in I$ and let $A = I \setminus \{x_0\}$.

A (inessential) *Laurent series* at x_0 is a function $g : A \to \mathbb{R}$ such that there exists an integer $k \in \mathbb{Z}$ and real number $a_k, a_{k+1}, \dots \in \mathbb{R}$ such that

$$g(x) = \sum_{n=k}^{\infty} a_k (x - x_0)^k.$$

If $k \geq 0$, a Laurent series is a power series.

Let $f : I \to \mathbb{R}$ be analytic. We attempt to find a Laurent series for $\frac{1}{f}$ at x_0. In particular, we try to find the number of negatively indexed coefficients in the inverse of f.

If we let $a_n = \frac{f^{(n)}}{n!}$, then

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We seek a function

$$g(x) = \sum_{n=k}^{\infty} b_n (x - x_0)^n,$$

where $k \leq 0$ and $b_k \neq 0$, such that $fg(x) = 1$ for every $x \in I$. Let c_n be the n^{th} term in the product; the lowest possible value for n is k. Then $c_n = \sum_{i-j=0} a_i b_j$;

when we multiply these series, we should get

$$\begin{aligned} c_k &= a_0 b_k \\ c_{k+1} &= a_0 b_{k+1} + a_1 b_k \\ c_{k+2} &= a_0 b_{k+2} + a_1 b_{k+1} + a_2 b_{k+3} \\ &\vdots \\ c_{-1} &= a_0 b_{-1} + \cdots + a_{k-1} b_k \\ c_0 &= a_0 b_0 + \cdots + a_k b_k \\ c_1 &= a_0 b_1 + \cdots + a_{k+1} b_k \end{aligned}$$

We want $c_0 = 1$ and all other $c_n = 0$. Then we better have $a_0 = 0$ (consider c_k), whence $a_1 = 0$ (considering c_{k+1}), and so forth up to a_{k-1}. The first a_n which is not equal to zero is at $n = k$.

13. Singularities

Let $I \subset \mathbb{R}$ be an open interval. Let $x_0 \in I$ and let $A = I \setminus \{x_0\}$. Let $g : A \to \mathbb{R}$ be analytic on A.

We say that g is *meromorphic* at x_0 if we may write g as an inessential Laurent series centered at x_0. We say that x_0 is a *singularity* of g.

Let $g : I \to \mathbb{R}$ be meromorphic at x_0. We say that the singularity at x_0 is *removable* if $\lim_{x \to x_0} g(x)$ exists; in this case, we may define

$$f(x) = \begin{cases} g(x) & \text{if } x \neq x_0; \\ \lim_{x \to x_0} f(x) & \text{if } x = x_0. \end{cases}$$

Then $f(x)$ is analytic at x_0; we think of f and g as interchangeable, and can write f as a power series around x_0.

We say that g has a *zero of order* n at x_0 if n is smallest integer such that the n^{th} coefficient of the Laurent expansion of f is nonzero. Equivalently, this is the maximum positive integer n such that $\frac{g(x)}{x^n}$ has a removable singularity at x_0.

We say that g has a *pole of order* n at x_0 if n is the minimum number of negatively indexed terms in the Laurent expansion of g. Equivalently, this is the maximum positive integer n such that $(x - x_0)^n g(x)$ has a removable singularity at x_0.

Note that g has a pole of order n at x_0 if and only if g has a zero of order $-n$ at x_0.

If f has a zero of order n at x_0 and g has a pole of order n at x_0, then fg has a removable singularity at x_0, and $fg(x_0) \neq 0$; equivalently, fg has a zero of order 0 at x_0.

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