HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC XIII
EULER AND COMPLEX NUMBERS

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Historical Background

Reference:
http://math.fullerton.edu/mathews/n2003/ComplexNumberOrigin.html.

Rafael Bombelli (Italian 1526 - 1572)
Recall that Cardano, in attempting to solve the cube equals cosa plus number case $x^3 = mx + n$, arrived at a negative sign under the radical. Tartaglia rebuked him, claiming that his methods were "totally false". Cardano, in attempting to go forward with this, eventually claimed that such considerations were "as subtle as they are useless".

However, in his 1572 treatise L'Algebra, Rafael Bombelli showed that roots of negative numbers have great utility indeed. Consider the depressed cubic $x^3 = 15x + 4$. Applying the method of Tartaglia and Cardano, we set $m = -15$ and $n = 4$. If $x = t - u$, then $3tu = m = -15$ and $t^3 - u^3 = n = 4$, so that $u^3 = -\frac{125}{t^3}$, and $t^3 + \frac{125}{t^3} = 4$. Then $t^6 - 4t^3 - 125 = 0$, and by the quadratic formula, $t^3 = 2 + \sqrt{-121} = 2 + 11\sqrt{-1}$, whence $u^3 = -2 + 11\sqrt{-1}$, and $x = \sqrt[3]{2 + 11\sqrt{-1}} - \sqrt[3]{-2 + 11\sqrt{-1}}$.

Now Bombelli, undeterred by the negative sign under the radical, wished to find a number whose cube was $2 + 11\sqrt{-1}$. Having a "wild thought", he assumed that such a number would be of the form $a + b\sqrt{-1}$. This produces

$$(a + b\sqrt{-1})^3 = (a^3 - 3ab^2) + (3a^2b - b^3)\sqrt{-1} = 2 + 11\sqrt{-1},$$

from which we conclude that $a^3 - 3ab^2 = 2$ and $3a^2b - b^3 = 11$. The first equation gives $a(a^2 - 3b^2) = 2$. Further assuming that a and b may be integers, and realizing that the only factors of 2 are 1 and 2, Bombelli discovered that $a = 2$ and $b = 1$ solved the first equation. Since they also solve the second, he found that $(2 + \sqrt{-1})^3 = 2 + 11\sqrt{-1}$. Thus $x = (2 + \sqrt{-1}) - (-2 + \sqrt{-1}) = 4$.

By considering $\sqrt{-1}$ as an acceptable quantity, Bombelli found a real solution to the cubic equation. This legitimized complex numbers as a legitimate area of study.

John Wallis (English 1619 - 1703)
Attempts to view complex solutions to quadratic equations as points on a plane.

Abraham de Moivre (French 1667 - 1754)
Used complex numbers in his formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

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Leonhard Euler (Swiss 1707 - 1783)
Understood DeMoivre's formula as giving solutions to the equation $x^n - 1 = 0$, viewed as vertices on a regular polygon.

Carl Friedrich Gauss (German 1777 - 1855)
Completed the geometric interpretation of the complex number $x + yi$ as the point (x, y) on the complex plane. Proved the Fundamental Theorem of Algebra.

Augustin-Louis Cauchy (French 1789 - 1857)
Formalized complex analysis and discovered many of its fascinating theorems.

1. Complex Algebra

Define addition and multiplication on the set $\mathbb{R}^2$ by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

Let $\mathbb{C}$ denote the set $\mathbb{R}^2$ together with this addition and multiplication; we call $\mathbb{C}$ the set of *complex numbers*.

Let $f : \mathbb{R} \to \mathbb{C}$ be given by $f(x) = (x, 0)$. This embeds the real line into $\mathbb{C}$, in a manner which preserves addition and multiplication; we call the image the *real axis*, and identify $\mathbb{R}$ with its image.

Let $i = (0, 1)$. Then $i^2 = i \cdot i = (-1, 0) = -1$. We call $\{(0, y) \mid y \in \mathbb{R}\}$ the *imaginary axis*.

Every element of $\mathbb{C}$ can be written as $x + iy$ in a unique way, where $x, y \in \mathbb{R}$; that is,

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i^2 = -1\}.$$

One can show that these operations have the following properties:

(F1) $a + b = b + a$ for every $a, b \in \mathbb{C}$;
(F2) $(a + b) + c = a + (b + c)$ for every $a, b, c \in \mathbb{C}$;
(F3) there exists $0 \in \mathbb{C}$ such that $a + 0 = a$ for every $a \in \mathbb{C}$;
(F4) for every $a \in \mathbb{C}$ there exists $b \in \mathbb{C}$ such that $a + b = 0$;
(F5) $ab = ba$ for every $a, b \in \mathbb{C}$;
(F6) $(ab)c = a(bc)$ for every $a, b, c \in \mathbb{C}$;
(F7) there exists $1 \in \mathbb{C}$ such that $a \cdot 1 = a$ for every $a \in \mathbb{C}$;
(F8) for every $a \in \mathbb{C} \smallsetminus \{0\}$ there exists $c \in \mathbb{C}$ such that $ac = 1$;
(F9) $a(b + c) = ab + ac$ for every $a, b, c \in \mathbb{C}$.

Together, these properties state that $\mathbb{C}$ is a *field*. Note that

- $0 = 0 + i0$;
- $1 = 1 + i0$;
- $-(x + iy) = -x + i(-y) = -x - iy$;
- $(x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}$.

2. Complex Geometry

Let $z = x + iy$ be an arbitrary complex number. The *real part* of z is $\Re(z) = x$. The *imaginary part* of z is $\Im(z) = y$. We view $\mathbb{R}$ as the subset of $\mathbb{C}$ consisting of those elements whose imaginary part is zero.

We graph complex number on the xy-plane, using the real part as the first coordinate and the imaginary part as the second coordinate. Under this interpretation, the set $\mathbb{C}$ becomes a real vector space of dimension two, with scalar multiplication given by complex multiplication by a real number. We call this vector space the *complex plane*.

Thus the geometric interpretation of complex addition is vector addition.

Let $z = x+iy$ be an arbitrary complex number. The *conjugate* of z is $\overline{z} = x - iy$. This is the mirror image of z under reflection across the real axis. The *modulus* of z is $|z| = \sqrt{x^2 + y^2}$. This is the length of z as a vector. Note that $z\overline{z} = |z|^2$. The *angle* of z, denoted by $\angle(z)$, is the angle between the vectors $(1, 0)$ and (x, y) in the real plane $\mathbb{R}^2$; this is well-defined up to a multiple of 2π.

Let $r = |z|$ and $\theta = \angle(z)$. Then $x = r\cos\theta$ and $y = r\sin\theta$. Define a function

$$\operatorname{cis} : \mathbb{R} \to \mathbb{C} \quad \text{by} \quad \operatorname{cis}(\theta) = \cos\theta + i\sin\theta.$$

Then $z = r\operatorname{cis}(\theta)$; this is the *polar representation* of z.

Recall the trigonometric formulae for the cosine and sine of the sum of angles:

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

and

$$\sin(A + B) = \cos A\sin B + \sin A\cos B.$$

Let $z_1 = r_1\operatorname{cis}(\theta_1)$ and $z_2 = r_2\operatorname{cis}(\theta_2)$. Then

$$\begin{aligned}
z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\
&= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\
&= r_1 r_2\operatorname{cis}(\theta_1 + \theta_2).
\end{aligned}$$

Thus the geometric interpretation of complex multiplication is:

(a) The radius of the product is the product of the radii;
(b) The angle of the product is the sum of the angles.

In particular, if $|z| = 1$, then $z = \operatorname{cis}(\theta)$ for some θ, and $z^n = \operatorname{cis}(n\theta)$. Restate this as

Theorem 1 (DeMoivre's Theorem)**.** $\operatorname{cis}^n(\theta) = \cos(n\theta) + i\sin(n\theta)$.

Example 1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 2z$. Then f dilates the complex plane by a factor of 2.

Example 2. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = iz$. Then f rotates the complex plane by 90 degrees.

Example 3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = (1 + i)z$. Note that $|1 + i| = \sqrt{2}$ and $\angle(1+i) = \frac{\pi}{4}$. Then f dilates the complex plane by a factors of $\sqrt{2}$ and rotates it by 45 degrees.

3. Complex Powers and Roots

Let $z = r\,\mathrm{cis}(\theta)$ and let $n \in \mathbb{N}$. Then $z^n = r^n\,\mathrm{cis}(n\theta)$.

The *unit circle* in the complex plane is

$$\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Note that if $u_1, u_2 \in \mathbb{U}$, then $u_1u_2 \in \mathbb{U}$.

Let $\zeta \in \mathbb{C}$ and suppose that $\zeta^n = 1$. We call ζ an n^{th} *root of unity.* If $\zeta^m \neq 1$ for $m \in \{1, \ldots, n-1\}$, we call ζ a *primitive* n^{th} *root of unity.*

Let $\zeta = \mathrm{cis}(\frac{2\pi}{n})$. Then $\zeta^n = \mathrm{cis}(n\frac{2\pi}{n}) = \mathrm{cis}(2\pi) = 1$; one sees that ζ is a primitive n^{th} root of unity. Thus primitive roots of unity exist for every n. As m ranges from 0 to $n-1$, we obtain distinct complex numbers ζ^m, all of which are n^{th} roots of unity. These are all of the n^{th} roots of unity; thus for each $n \in \mathbb{N}$, there are *exactly* n distinct n^{th} roots of unity.

If one graphs the n^{th} roots of unity in the complex plane, the points lie on the unit circle and they are the vertices of a regular n-gon, with one vertex always at the point $1 = 1 + i0$.

Let $z = r\,\mathrm{cis}(\theta)$. Then z has exactly n distinct n^{th} roots; they are

$$\sqrt[n]{z} = \sqrt[n]{r}\zeta_n^m\,\mathrm{cis}(\frac{\theta}{n}), \qquad \text{where } m \in \{0, \ldots, n-1\}.$$

The Fundamental Theorem of Algebra states that every polynomial with complex coefficients has a root in the complex numbers.

4. Complex Analysis

Let $f : \mathbb{C} \to \mathbb{C}$. We say that f is *continuous* at z_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

We say that f is *differentiable* at z_0 if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Complex differentiability has some amazing consequences; for example, it can be shown that every complex differentiable function is analytic.

We use the Taylor series expansion for several real transcendental functions in order to define their complex counterparts.

Define the complex exponential function

$$\exp : \mathbb{C} \to \mathbb{C} \text{ by } \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Define the complex sine function by

$$\sin : \mathbb{C} \to \mathbb{C} \text{ by } \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Define the complex cosine function by

$$\cos : \mathbb{C} \to \mathbb{C} \text{ by } \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

Note that exp, sin, and cos, when restricted to $\mathbb{R} \subset \mathbb{C}$, are defined so as to be consistent with other definitions of these real functions.

Define $\log : \mathbb{C} \to \mathbb{C}$ to be an inverse function of exp. Let $w, z \in \mathbb{C}$. We define w^z by

$$w^z = \exp(z \log(w)).$$

Thus $\exp(z) = e^z$.

Euler evaluated $\exp(iz)$, separating the real and imaginary parts, and found

$$\begin{aligned}
\exp(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= 1 + iz + i^2\frac{z^2}{2!} + i^3\frac{z^3}{3!} + i^4\frac{z^4}{4!} + i^5\frac{z^5}{5!} + i^6\frac{z^6}{6!} + i^7\frac{z^7}{7!} + \dots \\
&= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots) \\
&= \cos z + i \sin z.
\end{aligned}$$

In particular, if $z = \theta \in \mathbb{R}$, we have

Theorem 2 (Euler's Theorem)**.** *Let $\theta \in \mathbb{R}$. Then*

$$e^{i\theta} = \text{cis}(\theta).$$

Letting $\theta = \pi$, we get the beautiful

$$e^{i\pi} + 1 = 0,$$

a formula that relates the four most important constants in mathematics.

5. Sum of Square Reciprocals

5.1. Historical Background. Recall the *triangular numbers*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Leibnitz was challenged by Huygens to find the sum of their reciprocals. First factor out a 2 from all the terms $\frac{2}{n(n+1)}$; then compute

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \Big[\frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}\Big] \\ &= \sum_{n=1}^{\infty} \Big[\frac{1}{n} - \frac{1}{n+1}\Big] \\ &= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \dots \\ &= 1 - (\frac{1}{2} - \frac{1}{2}) - (\frac{1}{3} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{4}) - \dots \\ &= 1.\end{aligned}$$

Thus the sum of the reciprocals of the triangular numbers is 2.

Jacob Bernoulli, who knew that the harmonic series $\sum \frac{1}{n}$ diverges, then realized that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2.$$

Euler was able to compute the value to which the sum of the reciprocals of the square natural numbers converges.

5.2. Polynomials with Specified Roots. Let $a_1, \dots, a_n \in \mathbb{C}$. We wish to construct a canonical polynomial with these zeros. One way is to select the polynomial to be *monic*; that is, to have 1 as the leading coefficient. The polynomial with this property is just

$$f(x) = \prod_{i=1}^{n} (x - a_i).$$

In this case, we know that the coefficients of $f(x)$ are symmetric functions of the zeros. However, we may also choose to normalize the polynomial by selecting the constant coefficient to be 1. For this case, set

$$(\dagger) \qquad g(x) = \prod_{i=1}^{n} (1 - \frac{x}{a_i}).$$

The coefficient of x in $g(x)$ is

$$(*) \qquad \sum_{i=1}^{n} \frac{-1}{a_i}.$$

5.3. **Euler's Method.** Let $g(x) = \frac{\sin x}{x}$; the power series expansion for $g(x)$ is arrived at by taking the Taylor series for $\sin x$ and dividing it, term by term, by x, to obtain:

$$g(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots.$$

This has the appearance of a polynomial whose constant coefficient in 1, except that it infinitely many terms. Euler, being undeterred by this last fact, assumed that $g(x)$ could be written as an infinite product of linear terms as in equation (†).

Note that $g(0) = 1$; otherwise, the zeros of $g(x)$ are exactly those of $\sin x$; they are $Z = \{\pm\pi, \pm 2\pi, \pm 3\pi, \dots\}$. Thus Euler arrives at

$$\begin{aligned} g(x) &= \prod_{z \in Z} (1 - \frac{x}{z}) \\ &= \Big((1 - \frac{x}{\pi})(1 + \frac{x}{\pi})\Big)\Big((1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})\Big) \cdots \Big((1 - \frac{x}{n\pi})(1 + \frac{x}{n\pi})\Big) \cdots \\ &= \Big(1 - \frac{x^2}{\pi^2}\Big)\Big(1 - \frac{x^2}{4\pi^2}\Big) \cdots \Big(1 - \frac{x^2}{n^2\pi^2}\Big) \cdots \\ &= \prod_{n=1}^{\infty} \Big(1 - \frac{x^2}{n^2\pi^2}\Big). \end{aligned}$$

Multiplying out this infinite product, Euler finds the coefficient of the x^2 term, and equates it to the coefficient of the x^2 term of the power series expansion of $g(x)$, as in equation ($*$), to get

$$-\frac{1}{3!} = \sum_{n=1}^{\infty} \frac{-1}{n^2\pi^2}.$$

Multiply both sides by $-\pi^2$ to arrive at the mysterious result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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