MATH 3063 CALCULUS 1 — PROF. PAUL BAILEY — Project 1 — March 10, 2009 — SOLUTIONS

Problem 1. Let
$$f(x) = x^4 - 13x^2 + 36.$$

(a) Sketch a graph of f including all intercepts.

(b) Use the quadratic formula to find the solution set of $f(x) = k$, where $k \in \mathbb{R}$.

(c) Find conditions on k such that $f(x) = k$ has 0, 1, 2, 3, or 4 real solutions.

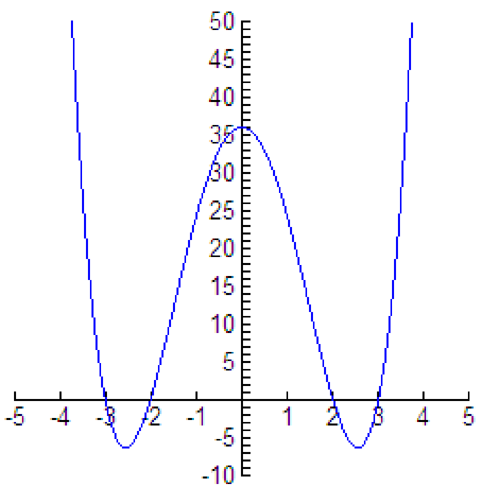
(d) Find $a \in \mathbb{R}$ such that $f(a) \leq f(x)$ for all $x \in \mathbb{R}$.

Solution.

(a) (Sketch the graph)

We factor f to get
$$f(x) = (x^2 - 4)(x^2 - 9) = (x + 3)(x + 2)(x - 2)(x - 3).$$

The y-intercept of f is $(0, 36)$ and the x-intercepts are $(-3, 0)$, $(-2, 0)$, $(2, 0)$, and $(3, 0)$. Since the leading coefficient of f is positive, we know that $f(x)$ is positive for $x > 3$. Moreover, since the multiplicities of the zeros are odd, we know that f changes sign at every zero. This produces the following graph:



(b) (Solve $f(x) = k$)

If $f(x) = k$, then $x^4 - 13x^2 + (36 - k) = 0$, so $x^2 = \frac{13 \pm \sqrt{169 - 4(36-k)}}{2}$, and
$$x = \pm\sqrt{\frac{13 \pm \sqrt{169 - 4(36 - k)}}{2}}.$$

(c) (Find conditions on k such that $f(x) = k$ has 0, 1, 2, 3, or 4 real solutions)
Let Δ denote the discriminant, so that $\Delta = 169 - 4(36 - k) = 25 + 4k$.

We know that the expression $x = \pm\sqrt{w}$ has

- 0 real solutions if $w < 0$
- 1 real solution if $w = 0$
- 2 real solutions if $w > 0$

Consider $x = \pm\sqrt{u \pm \sqrt{w}}$. Assuming (as in our case) that u is positive, Now

$$u - \sqrt{w} \geq 0 \Leftrightarrow u \geq \sqrt{w} \Leftrightarrow u^2 \geq w.$$

Thus the equation $x = \pm\sqrt{u \pm \sqrt{w}}$ has

(1) 2 real solutions if $w > u^2$ (since $u + \sqrt{w} > 0$)

(2) 3 real solutions if $w = u^2$ (since $u + \sqrt{w} > 0$ but $u - \sqrt{w} = 0$)

(3) 4 real solutions if $0 < w < u^2$

(4) 2 real solutions if $w = 0$

(5) 0 real solutions if $w < 0$

In our case, $u = \frac{13}{2}$ and $w = \frac{\Delta}{4}$; the constant denominators can successfully be ignored in our analysis. So we compute:

(1) $\Delta > 13^2 \Leftrightarrow 169 - 4(36 - k) > 169 \Leftrightarrow 36 - k < 0 \Leftrightarrow k > 36$, in which case there are 2 solutions. This is the part of the graph above the local maximum at $(0, 36)$, where a horizontal line intersects the graph in two points.

(2) $\Delta = 13^2 \Leftrightarrow k = 36$, in which case there are 3 solutions. This corresponds to the fact that the horizontal tangent line at the local maximum intersects the graph in three points.

(3) $0 < \Delta < 13^2 \Leftrightarrow k < 36$ and $25 + 4k > 0 \Leftrightarrow -\frac{25}{4} < k < 36$, in which case there are 4 solutions.

(4) $\Delta = 0 \Leftrightarrow k = -\frac{25}{4}$, in which case there are 2 solutions. This corresponds to the horizontal tangent through the local minima, the "bottom of the graph".

(5) $\Delta < 0 \Leftrightarrow k < -\frac{25}{4}$, in which case there are no real solutions.

Summarizing:

(1) if $k > 36$, there are 2 real solutions;

(2) if $k = 36$, there are 3 real solutions;

(3) if $-\frac{25}{4} < k < 36$, there are 4 real solutions;

(4) if $k = -\frac{25}{4}$, there are 2 real solutions;

(5) if $k < -\frac{25}{4}$, there are 0 real solutions.

(d) (Find a such that $(a, f(a))$ is a local minimum point)
We know that $f(a) = -\frac{25}{4}$, which occurs when $k = -\frac{25}{4}$ and $\Delta = 0$. Plugging $\Delta = 0$ into our solution for $f(x) = k$, we get

$$a = \pm\sqrt{\frac{13}{2}}.$$

$\square$

Problem 2. Let

$$f(x) = \frac{x^4 - 4x^2 + 3}{x^2 - 4}.$$

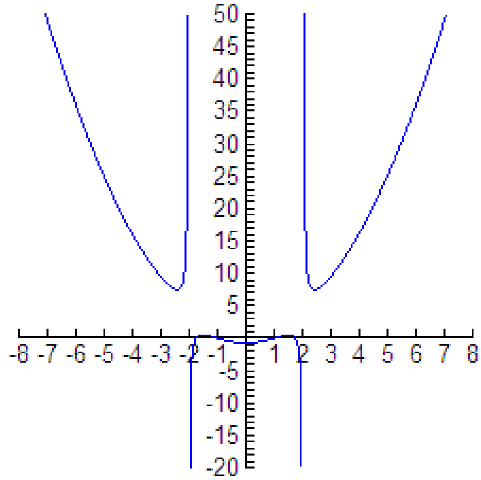
(a) Sketch a graph of f including all intercepts, vertical asymptotes, and the parabolic asymptote.

(b) Find conditions on k such that $f(x) = k$ has 0, 1, 2, 3, or 4 real solutions, where $k \in \mathbb{R}$.

Solution.

(a) (Sketch the graph)

The zeros are at $x = \pm 1, \sqrt{3}$, and the poles are at $x = \pm 2$. The the y-intercept is $(0, -\frac{3}{4})$, and the x-intercepts are $(\pm 1, 0)$, $(\pm\sqrt{3}, 0)$. The function f has vertical asymptotes at $x = \pm 2$ and the parabolic asymptote is $y = x^2$. This produces the following graph:



(b) (Find conditions on k which dictate the number of solutions to $f(x) = k$)
If $f(x) = k$, then $\frac{x^4-4x^2+3}{x^2-4} = k$, and multiplying by $x^2 - 4$ produces $x^4 - 4x^2 + 3 = k(x^2 - 4)$. We rearrange this in standard form to get

$$x^4 - (k+4)x^2 + (4k+3) = 0.$$

This is quadratic in x^2, and the quadratic formula followed by taking the square root gives

$$x = \pm\sqrt{\frac{(k+4) \pm \sqrt{(k+4)^2 - 4(4k+3)}}{2}}.$$

Set

$$\Delta = (k+4)^2 - 4(4k+3) = k^2 + 8k + 16 - 16k - 12 = k^2 - 8k + 4.$$

Again by the quadratic formula,

$$\Delta = 0 \quad \Leftrightarrow \quad k = \frac{8 \pm \sqrt{64 - 16}}{2} = 4 \pm 2\sqrt{3}.$$

Now

(1) $4 - 2\sqrt{3} < k < 4 + 2\sqrt{3}$ implies that $\Delta < 0$, in which case there are no real solutions;

(2) $k = 4 \pm 2\sqrt{3}$ implies that $\Delta = 0$, in which case there are two double solutions;

(3) $k = -\frac{3}{4}$ implies that $\Delta^2 = (k+4)$, in which case there are three solutions;

(4) $k > 4 + 2\sqrt{3}$, or $k < 4 - 2\sqrt{3}$ and $k \neq -\frac{3}{4}$, implies that $\Delta > 0$ and $(k+4) - \Delta \neq 0$, in which case the are four solutions.

These conditions dictate which horizontal lines are tangent to the graph; they are the lines $y = 4 + 2\sqrt{3}$ (two points of tangency), $y = 4 - 2\sqrt{3}$ (two points of tangency), and $y = -\frac{3}{4}$ (three intersection points, of which one is a point of tangency). $\square$

Problem 3. Let
$$f(x) = x^3 - x \quad \text{and} \quad g(x) = x^3 + 3x^2 + 2x + 6.$$

(a) Solve $f(x) = 0$ by factoring.

(b) Sketch the graph of f.

(c) Compute $f(x+1)$.

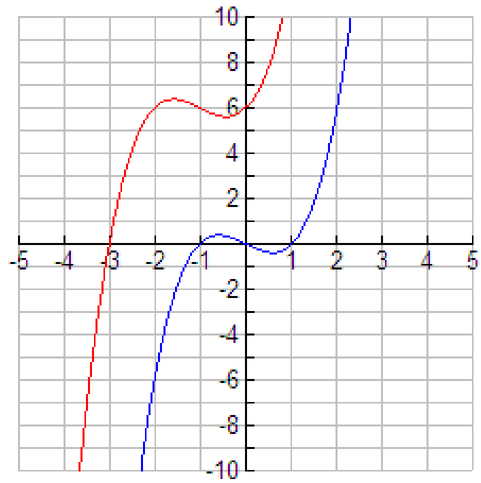
(d) Describe how the graph of g can be obtained from the graph of f by transformations.

(e) Sketch the graph of $g(x)$ (including the x-intercept).

Solution.

(a) (Solve $f(x) = 0$)
We have $f(x) = x(x-1)(x+1)$, so $f(x) = 0$ if and only if $x = -1, 0, 1$.
(c) (Compute $f(x+1)$)
We have
$$f(x+1) = (x+1)^3 - (x+1) = (x^3 + 3x^2 + 3x + 1) - (x+1) = x^3 + 3x^2 + 2x.$$

(d) (Get g from f)
We see that $g(x) = x^3 + 3x^2 + 2x + 6 = f(x+1) + 6$, so the graph of g is the graph of f shifted left 1 and up 6.
(e) (Sketch $g(x)$) Since the leading coefficient of f is positive and the each zero has multiplicity one, the graph is easily obtained. Now shift it.



$\square$

Problem 4. Consider the rational function

$$f(x) = \frac{x^2 - x}{x - 2}.$$

This function has a unique local maximum at a on the interval $(0, 1)$ (i.e., a is the x-coordinate of the highest point on the graph between $x = 0$ and $x = 1$).

(a) Find the zeros, poles, intercepts, and asymptotes of f, and sketch the graph of f.

(b) Find all values for k such that the equation $f(x) = k$ has a unique real solution.

(c) Find $f(a)$.

(d) Find a.

Solution. We have $f(x) = \frac{x^2-x}{x-2}$, so $f(x) = k$ if and only if $x^2 - x = kx - 2k$, or

$$x^2 - (k + 1)x + 2k = 0.$$

Then

$$x = \frac{(k + 1) \pm \sqrt{(k + 1)^2 - 8k}}{2}.$$

If $\Delta = (k + 1)^2 - 8k = k^2 - 6k + 1 = 0$, then

$$\Delta = 0 \Rightarrow = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.$$

So $f(x) = k$ has exactly one solution when $k = 3 \pm 2\sqrt{2}$; from the graph we see that the local maximum between $x = 0$ and $x = 1$, when $k = f(a) = 3 - 2\sqrt{2}$. This occurs when $a = 2 - \sqrt{2}$. $\square$