Math 1525 Calculus I Project 2 Solutions
Prof. Paul Bailey April 13, 2009

Problem 1. Let $f(x) = x^2 + 9$. Find the equation of the unique line through the origin with positive slope which is tangent to the graph of f.

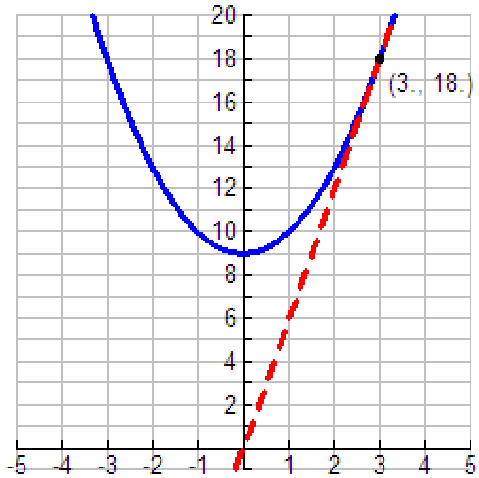
Solution. Let $a \in \mathbb{R}$ denote the x-coordinate of the point of tangency. Equating the derivative to the standard definition of slope, we see that the slope of the line is

$$m = \frac{f(a)}{a} = 2a.$$

Thus

$$\frac{x^2 + 9}{a} = 2a \Rightarrow x^2 + 9 = 2a^2 \Rightarrow a^2 = 9 \Rightarrow a = 3.$$

The slope, then, is $m = 6$, and the line is $y = 6x$.



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Problem 2. Let $f(x) = 1 - x^2$ and $g(x) = (x-2)^2$. Find the equation of the unique line which is tangent to the graphs of f and g.

Solution. Let $(a, f(a))$ denote the point of tangency on the graph of f, and let $(b, g(b))$ denote the point of tangency on the graph of g. Equating the derivative to the standard definition of slope, we see that the slope of the line is

$$m = \frac{g(b) - f(a)}{b - a} = -2a = 2b - 4.$$

From the second equation, $b = 2 - a$, so

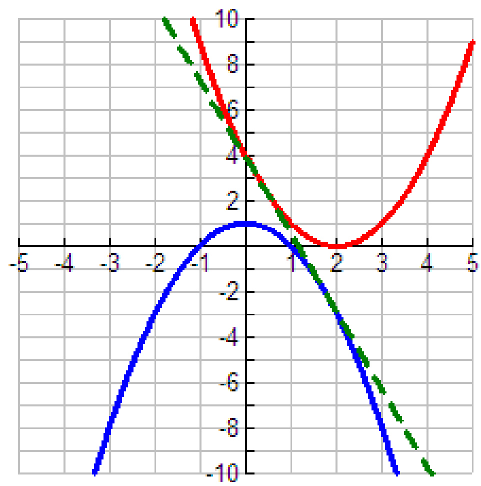
$$\begin{aligned} \frac{(b-2)^2 - (1-a^2)}{b-a} = -2a \Rightarrow \frac{(-a)^2 - (1-a^2)}{(2-a)-a} &= -2a \\ \Rightarrow \frac{2a^2 - 1}{2 - 2a} &= -2a \\ \Rightarrow 2a^2 - 1 &= 4a^2 - 4a \\ \Rightarrow 2a^2 - 4a + 1 &= 0 \\ \Rightarrow a &= \frac{4 \pm \sqrt{16 - 8}}{4} \\ \Rightarrow a &= 1 \pm \frac{\sqrt{2}}{2}. \end{aligned}$$

This gives two solutions; the line graphed below has $a = 1 + \frac{\sqrt{2}}{2}$, so that $b = 1 - \frac{\sqrt{2}}{2}$. Thus $m = -2a = -2 - \sqrt{2}$, and the line is

$$y = (-2a)(x-a) + f(a) = -2ax + 2a^2 + (1 - a^2) = -2ax + (a^2 + 1);$$

that is,

$$y = (-2 - \sqrt{2})x + \frac{5}{2} + \sqrt{2}.$$



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Problem 3. Find the area of a circle centered at the origin which is tangent to the parabola $y = 1 - x^2$.

Solution. The line from the center of a circle to a point on the circle is perpendicular to the tangent line at that point. Let m be the slope of the tangent line, and let $(a, f(a))$ be the point of tangency, where $a > 0$. Then

$$m = f'(a) \quad \text{and} \quad -\frac{1}{m} = \frac{f(a)}{a},$$

where $f(x) = 1 - x^2$. So

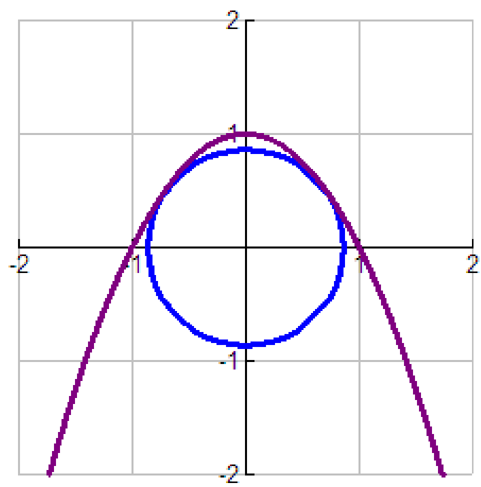
$$f'(a) = -\frac{a}{f(a)} \Rightarrow -2a = -\frac{a}{1-a^2} \Rightarrow 2a - 2a^3 = a \Rightarrow 2a^3 = a \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \frac{\sqrt{2}}{2}.$$

Now $f(a) = f(\frac{\sqrt{2}}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$. The radius of the circle is

$$r = \sqrt{(\frac{\sqrt{2}}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2} + \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

Thus the area of the circle is

$$A = \pi r^2 = \frac{3\pi}{4}.$$



$\square$

Problem 4. Let $f(x) = cx - x^3$, where $c \in \mathbb{R}$ is positive. Then there exist $a, b \in \mathbb{R}$ with $a < b$ such that f has a local minimum at $x = a$ and a local maximum at $x = b$.

Let m be the slope of the line through $(a, f(a))$ and $(b, f(b))$. Find c such that $m = 1$.

Solution. Since the leading coefficient of f is negative, we know that $a < b$.

We have $f(x) = x(c - x^2)$, so f has zeros at 0 and $\pm\sqrt{c}$. Also, $f'(x) = c - 3x^2$, so $f'(x) = 0$ when $x = \pm\sqrt{\frac{c}{3}}$. Since f has a local maximum at b, the derivative is zero there, so

$$\boxed{c - 3b^2 = 0.}$$

Thus $b = \sqrt{\frac{c}{3}}$ and $a = -b$. We re

Now the slope of the line through the local extreme points is

$$m = \frac{f(b) - f(a)}{b - a} = \frac{2f(b)}{2b} = \frac{f(b)}{b} = 1 \Rightarrow f(b) = b \Rightarrow cb - b^3 = b;$$

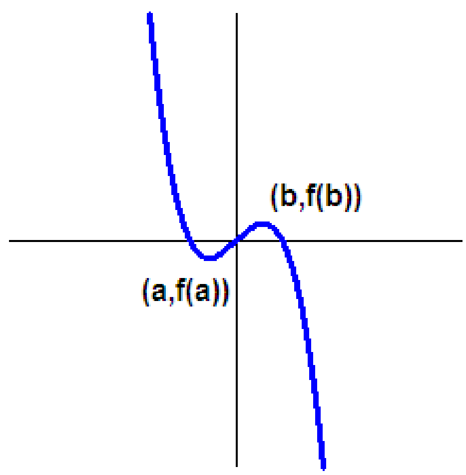
therefore

$$\boxed{c - b^2 = 1.}$$

From the first boxed equation, $c = 3b^2$, and plugging this into the second boxed equation gives

$$3b^2 - b^2 = 1 \Rightarrow 2b^2 = 1 \Rightarrow b = \frac{\sqrt{2}}{2}.$$

Thus $c = 3b^2 = \frac{3}{2}$.



$\square$

Problem 5. Let $a, b, c \in \mathbb{R}$ with $a < b < c$, and let f be a cubic polynomial with zeros at a, b, and c. The average of the zeros is

$$w = \frac{a+b+c}{3}.$$

Show that f has an inflection point at $(w, f(w))$.

Solution. Since vertically stretching the graph of a function does not change the x-coordinate of its extreme points, without loss of generality, we may assume that the leading coefficient of f is 1. Then

$$f(x) = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc.$$

Thus

$$f'(x) = 3x^2 - 2(a+b+c)x + (ab+ac+bc),$$

and

$$f''(x) = 6x - 2(a+b+c).$$

If $f''(w) = 0$, then $6w = 2(a+b+c)$, so

$$w = \frac{a+b+c}{3}.$$

Now w is a zero of $f''(x)$ of multiplicity one, so the sign of $f''(x)$ does change at w, which implies that w is a point of inflection. $\square$