This is a *group quiz*: please discuss the problems amongst yourselves and come up to the best solution. Then write what you think is the best solutions on this page. Explain your results.

Problem 1. In seven card stud, you are dealt seven cards from a standard deck. Find the probability that at least two cards have the same rank.

Solution. There are

$$\binom{52}{7} = 133784560$$

possible hands of seven cards. To construct a hand without a pair, we select any card (52 choices), then remove all cards of that rank, then select another card (48 choices), and so forth; after obtaining seven cards, we have the number of ways of selecting such a hand *where the order matters*, and we must divide by the number of ways of rearranging the cards (7!) to kill the order. This is equal to selecting seven ranks out of 13, and multiplying by the number of cards in each rank:

$$\frac{52 \times 48 \times 44 \times 40 \times 36 \times 32 \times 28}{7!} = \binom{13}{7} \times 4^7 = 1716 \times 16304 = 28114944.$$

So the probability of *not* receiving a pair is

$$P(\text{no pair}) = \frac{28114944}{133784560} \approx 0.21015.$$

Therefore, the probability of receiving a pair is

$$P(\text{pair}) \approx 1 - 021015 = 0.78985 \approx 79\%.$$

□

Problem 2. In five card draw, you are dealt five cards, exactly two of which are aces. You keep the two aces, and discard the rest, and receive three more cards. Find the probability that the resulting hand contains at least three aces.

Solution 1: via counting. Our situation is that we have a deck of 47 cards containing two aces. There are

$$\binom{47}{3} = 16215$$

ways to receive three cards from this deck. How many sets of three cards have no aces? There are 45 cards which are not aces, so there are

$$\binom{45}{3} = 14190$$

sets of three cards which do not contain a ace. Therefore, the probability of not receiving an ace is

$$P(\text{no ace}) = \frac{14190}{16215} \approx 0.87512.$$

Thus the probability of being dealt at least one ace is

$$P(\text{ace}) \approx 1 - 0.87512 = 0.12488 \approx 12.5\%.$$

□

Solution 2: via conditioning. We are dealt three cards; our sample space S consists of ordered triples of three distinct cards. Let E be the event of being dealt at least one ace and let E_i denote the event of being dealt an ace on the i^{th} card. Then $E = E_1 \cup E_2 \cup E_3$; however, these events overlap. To partition E, we write

$$E = E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap (E_1^c \cap E_2^c)),$$

which express E as the disjoint union of these three sets. Thus the probability of E is the sum of the probabilities of these events. Now

$$\begin{aligned}
P(E) &= P(E_1) + P(E_2 \cap E_1^c) + P(E_3 \cap (E_1^c \cap E_2^c)) \\
&= P(E_1) + P(E_1^c)P(E_2 \mid E_1^c) + P(E_1^c \cap E_2^c)P(E_3 \mid E_1^c \cap E_2^c) \\
&= P(E_1) + P(E_1^c)P(E_2 \mid E_1^c) + P(E_1^c)P(E_2^c \mid E_1^c)P(E_3 \mid E_1^c \cup E_2^c) \\
&= \frac{2}{47} + \frac{45}{47} \cdot \frac{2}{46} + \frac{45}{47} \cdot \frac{44}{46} \cdot \frac{2}{45}
\end{aligned}$$

□

Problem 3. You and another poker player are each dealt full houses (three cards of one rank and two of another). Your's contains 3 eights and 2 aces. Find the probability that your hand is better than his.

Solution. Our situation is that we have a deck of 47 cards which consists of the standard 52 cards with three eights and two aces missing. From this, we know that a full house has been dealt. Such a full house must contain three cards of a rank other than eights or aces. Fixing such a rank, there are four choose three possible ways to selecting the three cards out of four. For the other pair, there are four choose two possible ways of selecting a pair of a given rank other than eights, the three of a kind rank, and aces; there is one pair of aces remaining. Thus the number of possible hands of a given three of a kind rank is

$$N = \binom{4}{3} \times (10 \times \binom{4}{2} + 1).$$

Losing or winning is completely dependent on the three of a kind rank. If it is two through eight, you win. If it is nine through king, you loose. There are no other possibilities. So the number of winning hands is $6N$ and the number of loosing hands is $5N$. Therefore,

$$P(\text{win}) = \frac{6N}{6N + 5N} = \frac{6}{11} = 0.\overline{54} \approx 55\%.$$

□