EUCLIDEAN SPACES AND VECTORS

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1. Introduction

Our ultimate goal is to apply the techniques of calculus to higher dimensions. We begin by discussing what mathematical concepts describe these higher dimensions.

Around 300 B.C. in ancient Greece, Euclid set down the fundamental laws of *synthetic geometry*. Geometric figures such as triangles and circles resided on an abstract notion of *plane*, which stretched indefinitely in two dimensions; the Greeks also analysed solids such as regular tetrahedra, which resided in *space* which stretched indefinitely in three dimensions.

The ancient Greeks had very little algebra, so their mathematics was performed using pictures; no *coordinate system* which gave positions to points was used as an aid in their calculations. We shall refer to the uncoordinatized spaces of synthetic geometry as *affine spaces*. The word affine is used in mathematics to indicate lack of a specific preferred origin.

The notion of coordinate system arose in the *analytic geometry* of Fermat and Descartes after the European Renaissance (circa 1630). This technique connected the algebra which was florishing at the time to the ancient Greek geometric notions. We refer to coordinatized lines, planes, and spaces as *cartesian spaces*; these are composed of *ordered n-tuples* of real numbers.

Since affine spaces and cartesian spaces have essentially the same geometric properties, we refer to either of these types of spaces as *euclidean spaces*.

Just as coordinatizing affine space yields a powerful technique in the understanding of geometric objects, so geometric intuition and the theorems of synthetic geometry aid in the analysis of sets of n-tuples of real numbers.

The concept of *vector* will be the most prominent tool in our quest to use differntial calculus in higher dimensional spaces. Vectors may be defined and manipulated entirely in the geometric realm or entirely algebraically.

Our goal is to define "vector" and various vector operations both geometrically and algebraically and to show that these definitions are in agreement. Specifically, we will construct a correspondence

$$\{\text{equivalence classes of arrows in affine space}\} \longleftrightarrow \{\text{points in cartesian space}\}$$

which respects vector operations; for example, the geometric sum of two equivalence classes of arrows will correspond to the algebraic sum of the corresponding points.

In order to make the above ideas precise, we will use the language of sets.

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2. Basic Set Concepts

A *set* is an aggregate of *elements*; the elements are said to be *contained* in the set. A set is determined by the elements it contains. That is, two sets are considered equal if and only if they contain the same elements.

If A is a set and x is an element contained in A, this relationship is denoted $x \in A$. If y is not in A, we may write $y \notin A$.

If A and B are sets and all of the elements in B are also contained in A, we say that B is a *subset* of A or that B is *included* in A and write $B \subset A$.

A set containing no elements is called the *empty set* and is denoted $\varnothing$. Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

Let A and B be subsets of some "universal set" U and define the following set operations:

$$\begin{aligned} \text{Intersection:} \quad & A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\} \\ \text{Union:} \quad & A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\} \\ \text{Complement:} \quad & A \smallsetminus B = \{x \in U \mid x \in A \text{ and } x \notin B\} \end{aligned}$$

A picture corresponds to each of these. Such picture are called *Venn diagrams*.

Example 1. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5\}$. Then $A \cap B = \{1, 3, 5\}$, $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$, $A \smallsetminus B = \{7, 9\}$, and $B \smallsetminus A = \{2, 4\}$. □

Example 2. Let $C = [1, 5] \cup (10, 16)$ and let $\mathbb{N}$ be the set of counting numbers. How many elements are in $C \cap \mathbb{N}$?

Solution. The set $C \cap \mathbb{N}$ is the set of natural numbers between 1 and 5 inclusive and between 10 and 16 exclusive. Thus $C \cap \mathbb{N} = \{1, 2, 3, 4, 5, 11, 12, 13, 14, 15\}$. Therefore $C \cap \mathbb{N}$ has 10 elements. □

Example 3. Use Venn diagrams to find an alternate expression for each of the following:

(a) $(A \cup B) \cap (A \cup C)$ $[= A \cup (B \cap C)]$;
(b) $(A \smallsetminus B) \cup (B \smallsetminus A)$ $[= (A \cup B) \smallsetminus (A \cap B)]$.

The concepts of intersection and complement arise naturally in single variable calculus. If $g(x)$ and $h(x)$ are real-valued functions of a real variable and $\mathbb{R}$ is the set of real numbers, then

$$\operatorname{dom}\left(\frac{g(x)}{h(x)}\right) = (\operatorname{dom}(g(x)) \cap \operatorname{dom}(h(x))) \smallsetminus \{x \in \mathbb{R} \mid h(x) = 0\}.$$

Example 4. The domain of the function

$$f(x) = \frac{\sqrt{4 - x^2}}{\log x^3}$$

is $([-2, 2] \cap (0, \infty)) \smallsetminus \{1\} = (0, 1) \cup (1, 2]$. □

3. Standard Sets of Numbers

The following sets of numbers are standard:

Natural Numbers:	$\mathbb{N} = \{1, 2, 3, \dots\}$
Integers:	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational Numbers:	$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}\}$
Real Numbers:	$\mathbb{R} = \{\text{Dedekind cuts in } \mathbb{Q}\}$
Complex Numbers:	$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$

Thus $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

The following standard notation gives subsets of the real numbers, called *intervals*:

$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$	(finite open)
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$	(finite open)
$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$	
$(a, b] = \{x \in \mathbb{R} \mid a < x \le b\}$	
$(-\infty, b] = \{x \in \mathbb{R} \mid x \le b\}$	(infinite closed)
$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$	(infinite open)
$[a, \infty) = \{x \in \mathbb{R} \mid a \le x\}$	(infinite closed)
$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$	(infinite open)

We define a subset of the real numbers to be *open* if and only if it is the union of open intervals. A subset of the real numbers is called *closed* if and only if its complement is open.

The rest of this section is intended to loosely describe why we use the real numbers in geometry as opposed to others sets of numbers. Don't be concerned if it seems confusing, but read it to see if it gives you a feel for the idea of continuum.

The first three of these sets have an algebraic nature. The natural numbers allow us to count, add, and multiply. The integers were developed from the natural numbers to allow subtraction. The rational numbers were developed from the integers to allow division.

In the definition of rational numbers, the notation $\frac{p}{q}$ indicates the *equivalence class* of the rational number represented by $\frac{p}{q}$. Here we say that two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if $ad = bc$; we wish these two fractions to represent the same rational number. We will use the idea of equivalence in our definition of vectors.

The real numbers were developed for geometric and analytic reasons; they are an *ordered continuum.* Here continuum means that all sequences which become indefinitely close in their tails converge to a number in the set. Note that this is not true for the rational numbers; the sequence

$$\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1414213, \dots\}$$

consists of rational numbers but converges to $\sqrt{2}$, which is not a rational number. The rational number line has "holes" where the irrational numbers belong, and for this reason it does not model the synthetic notion of an affine line as well as the real numbers.

Dedekind cuts are separations of the rational numbers into two ordered sets such that everything in one of the sets is less than everything in the other, and such that the set of larger numbers never contains a smallest number. Thus if we "cut" at a rational, we put that in the set of smaller numbers, and if we "cut" at a "hole", we have detected an irrational number. The set of such "cuts" has no "holes".

The complex numbers were developed from the real numbers so that all polynomials may be factored. However, the complex numbers cannot be ordered in a way that preserves their algebraic properties; for this reason, we primarily use the real numbers for analytic geometry, keeping in mind that these other sets of numbers serve useful purposes which we shall exploit whenever it is advantageous.

4. Cartesian Product

Let $a, b, c, d \in U$. An *ordered pair* (a, b) is defined by the property

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

The *cartesian product* of the sets A and B is defined by $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Similarly, we have *ordered triples* (a,b,c) and the cartesian product of three sets $A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$. We continue with *ordered n-tuples* and the cartesian product of n sets. If A is a set, the cartesian product of A with itself n times is denoted A^n. For example, $A^2 = A \times A$ and $A^3 = A \times A \times A$. The entries of an ordered n-tuple in such a cartesian product are called *coordinates.*

Proposition 1. *Let A, B, C, and D be sets. Then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Similar statements are true for higher cartesian products.*

Example 5. Let $A = [1, 3] \times [2, 4) \times (3, 5)$. How many elements are in the set $A \cap \mathbb{Z}^3$?

Solution. We have $B = [1, 3] \cap \mathbb{Z} = \{1, 2, 3\}$, $C = [2, 4) \cap \mathbb{Z} = \{2, 3\}$, and $D = (3, 5) \cap \mathbb{Z} = \{5\}$. Then

$$A \times \mathbb{Z}^3 = B \times C \times D = \{(1, 2, 5), (1, 3, 5), (2, 2, 5), (2, 3, 5), (3, 2, 5), (3, 3, 5)\},$$

a set with 6 elements. □

If the set A is ordered, such as the case where $A = \mathbb{R}$, we may graph ordered pairs and sets of order pairs by drawing perpendicular lines, called axes, which are "ruled-off". Each line represents a copy of the real numbers, and an ordered pair is plotted as the appropriate point. By convention, the horizontal axis is designated x and represents the first coordinate, and the vertical axis is designated y and represents the second coordinate. For example, the graph of the set $[0, 1] \times [1, 2]$ is a square which touches the y-axis and is lifted 1 unit above the x-axis. Note that the graph of a function f is the graph of the set $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$.

We may also graph ordered triples of real numbers on a flat piece of paper, using perspective to give the illusion of depth. In this case, tradition demands that the first coordinate of an ordered triple is labeled x, the second y, and the third z; and that the positive z-axis points north, the positive y-axis points east, and the positive x-axis points southwest so that it appears to emanate from the page. Points and sets are plotted against this coordinate system in the natural way.

Example 6. Let $A = [1, 3] \times [2, 4) \times (3, 5)$. Graph the set $A \cap \mathbb{Z}^3$.

Solution. We have seen that

$$A \cap \mathbb{Z}^3 = \{(1,2,5),(1,3,5),(2,2,5),(2,3,5),(3,2,5),(3,3,5)\}.$$

Graph these points. □

Example 7. Draw the box with diagonal vertices $P(1,1,2)$ and $Q(4,-1,4)$.

Solution. First we find the other six vertices. These are $(4,1,2)$, $(4,-1,2)$, $(1,-1,2)$, $(1,-1,4)$, $(4,1,4)$, and $(1,1,4)$. Graph these and draw the edges which move parallel to a coordinate axis. □

5. Distance

We wish to define the *distance* between to points in $\mathbb{R}^n$ in such a way that it will agree with our geometric intuition into the pictures produced by our graphs. Here we use the Pythagorian Theorem.

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

In particular, we call the distance between a point and the origin the *norm* of the point. Thus if $P = (x, y, z)$,

$$|P| = \sqrt{x^2 + y^2 + z^2}.$$

Other names for this quantity include *modulus*, *magnitude*, *absolute value*, or *length* of the point.

Example 8. The distance between $(2,5,-1)$ and $(-4,3,8)$ is

$$d = \sqrt{(-4-2)^2 + (3-5)^2 + (8-(-1))^2} = \sqrt{36+4+9} = \sqrt{49} = 7.$$

□

6. Equations

We may consider subsets of $\mathbb{R}^n$ such that the coordinates of the points in the subset are related in some specified way. The common way of doing this is to consider *equations* with the coordinates as *variables*. The set of all points which, when their coordinates are plugged into the equation cause the equality to be true, is called the *solution set*, or *locus* of the equation.

Consider the solution set in $\mathbb{R}^3$ of the equation $z = 0$. This is the set of points of the form $(x, y, 0)$. This set is immediately identified with $\mathbb{R}^2$ in the natural way. This set is called the xy-plane. Similarly, the solution sets of $x = 0$ and $y = 0$ are called the yz-plane and the xz-plane, respectively. Together, these sets are called *coordinate planes*.

Example 9. Find the locus of the equation $xyz = 0$.

Solution. If $xyz = 0$, either $x = 0$, $y = 0$, or $z = 0$. Thus the solution set is the union of the solution sets for these latter equation; that is, the locus of the equation $xyz = 0$ is the union of the coordinate planes. □

Example 10. Find an equation whose solution set is the union of the coordinate axes.

Solution. The x-axis is the set of points where $y = 0$ and $z = 0$. We can acheive the x-axis as the solution set of $y^2 + z^2 = 0$. Thus we can see that the solution set of

$$(x^2 + y^2)(x^2 + z^2)(y^2 + z^2) = 0$$

is the union of the coordinate axes. □

Now consider sets of points which simultaneously satisfy all of the equations in a collection of equations. Such sets are merely the intersection of the solution sets. For example, the solution set of $\{x = 0, y = 0\}$ is the z-axis.

If one of the variables is missing from an equation, its locus in $\mathbb{R}^3$ is a *curtain* (or *cylinder*), because the third variable can be anything.

Example 11. The locus in $\mathbb{R}^2$ of the equation $y = 2x + 1$ is a line, but in $\mathbb{R}^3$ it is a plane. The locus in $\mathbb{R}^3$ of the equation $z = \sin y$ is a rippled "plane"; any point of the form $(x, y, \sin y)$ is in the locus.

Let $P_0 = (x_0, y_0, z_0)$ be some fixed point in $\mathbb{R}^3$ and let $r \in \mathbb{R}$. Consider the equation $d(P, P_0) = r$, where $P = (x, y, z)$ is a variable point. The solution set of this equation is exactly the set of all points in $\mathbb{R}^3$ whose distance from P_0 is equal to r. This set is called the *sphere of radius r centered at P_0*. Since distance is always positive, we may square both sides of the equation and obtain a new equation with the same solution set. Thus the equation of a sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Example 12. Find the radius and center of the sphere given by $x^2 + y^2 + z^2 + 6x - 16 = 0$.

Solution. Complete the square. The locus of the above equation is the same as the locus of $x^2 + 6x + 9 + y^2 + z^2 = 16 + 9$, i.e., $(x + 3)^2 + y^2 + z^2 = 25$. Thus the center is $(-3, 0, 0)$ and the radius is 5. □

7. Coordinatization

In order to apply the techniques of analytic geometry to synthetic geometry or to a real-life problem, we must impose a coordinate system on affine space.

To do this in three dimensions, we first select a point in affine space and calling it the origin. We then select three perpendicular lines that intersect at the origin as the axes. We must also select, on each axis, one of the two directions as the positive direction. By convention, this is done in such a way that the ordered system of axes constitute a right-handed orientation. We use the "right-hand rule": with your right hand, make a fist, let your thumb point up and your point your index finger out, parallel to your arm. Let your middle finger stick out perpendicular to your index finger. Then your axes should be oriented such that the index finger points in the positive x direction, your middle finger points in the positive y direction, and your thumb points in the positive z direction.

Now the coordinates of a point are given by the signed distance of that point to the corresponding coordinate plane. No two points occupy the exact same location, so each point has its own unique coordinates. This process is called a *coordinatization* of affine space.

Coordinatizing an affine space gives us a cartesian space. These spaces have essentially the same properties. The reason for the distinction is to help us keep in mind that we may often select the coordinate system which best suits our needs in a particular problem.

8. Equivalence Classes

Since vectors will be defined as equivalence classes of arrows, here is a brief description of this idea.

The elements of a set may themselves be sets. For example, the set of all major league baseball teams may be considered to be composed of sets of players, each player the set of molecules which comprise him, each molecule a set of subatomic particles, et cetera.

Two sets A and B are called *disjoint* if $A \cap B = \varnothing$. A *partition* of a set X is a collection $\mathcal{C}$ of subsets of X such that the sets in $\mathcal{C}$ are mutually disjoint and the union of the sets in $\mathcal{C}$ is X. Each member of $\mathcal{C}$ is called and *equivalence class*. A member of an equivalence class is called a *representative* of that class. The relationship between two members of the same class is called an *equivalence relation*.

A familiar example of this is the set of rational numbers. Let $A = \mathbb{Z} \times \mathbb{Z}$, the set of ordered pairs of integers. We think of the ordered pair (a, b) as representing the faction $\frac{a}{b}$. We define an equivalence relation on A by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

Thus $(1, 2) \sim (2, 4)$, etc. This is how the rational numbers are constructed.

9. Affine Space

We consider four types of affine space: the point, line, plane, and space of synthetic geometry, corresponding to the dimensions 0, 1, 2, and 3 respectively. The phrase "Affine space of dimension n" means the set of points in n dimensional space of synthetic geometry. Affine spaces come equipt with the notions of distance and angle.

Let P, Q, and R be three points in affine space. We will use the following notation:

- PQ is the line through P and Q;
- $\bar{PQ}$ is the line segment between P and Q;
- $|PQ|$ is the distance between P and Q;
- $\angle PQR$ is the angle between the line segments $\bar{PQ}$ and $\bar{QR}$;
- $\triangle PQR$ is the triangle with sides $\bar{PQ}$, $\bar{QR}$, and $\bar{RP}$.

10. Arrows and Vectors

An *arrow* in affine space is a directed line segment; that is, it is a line segment with one of the ends specified as the "beginning" and the other specified as the "end". The beginning point is called the *tail* of the arrow and the ending point is called the *tip* of the arrow.

If P and Q are points in affine space, the arrow from P to Q is denoted $\hat{PQ}$. If $P = Q$, the arrow $\hat{PQ} = \hat{PP}$ is called the *zero arrow at* P.

The set of all arrows in affine n-space is denoted $\mathcal{A}^n$.

An arrow is determined by

(1) *position*, which is determined by its tail;
(2) *magnitude*, which is the distance between the tip and the tail;
(3) *direction*, which is the direction of the ray from the tail to the tip, and is determined by the line through the points and side of the line on which the tip lies relative to the tail.

Note that the zero arrows have no direction.

Suppose that two nonzero arrows lie on the same line. We say that they have the *same orientation* if they point in the same direction; otherwise they have the *opposite orientation*.

Notice that for every arrow $\hat{RS}$ and for every point P, there is another arrow $\hat{PQ}$ whose tail is at P which has the same magnitude and direction as $\hat{RS}$.

We wish to consider only the magnitude and direction attributes of arrows, and ignore the position. This allows use to "slide" arrows around in affine space, and consider them to start at the tail or at the tip of some other arrow. To do this, we call two arrows *equivalent* if they have the same direction and magnitude.

A *vector* is an equivalence class under this equivalence relation. That is, a vector is unpositioned direction and length. Suppose that P, Q, R, and S are points in affine space. Suppose additionally that the direction and length of $\hat{PQ}$ is the same as the direction and length of $\hat{RS}$. Then $\hat{PQ}$ and $\hat{RS}$ are said to *represent* the same vector, and this vector may be denoted $\vec{PQ}$ or $\vec{RS}$.

The set of all vectors in affine n-space is denoted $\mathcal{V}^n$.

The *length* of a vector is the length of any arrow that represents it. If $\vec{v}$ is a vector, its length is denoted $|\vec{v}|$. Notice that there is exactly one equivalence class of arrows which contains all of the zero arrows. This equivalence class is called the *zero vector*.

Now suppose that we have coordinatized affine space. Then each vector has exactly one representative which is an arrow whose tail is at the origin. Such an arrow is said to be in *standard position*. The tip of this arrow is a point in $\mathbb{R}^n$. Each vector corresponds to exactly one point in $\mathbb{R}^n$ in this way.

If $P \in \mathbb{R}^n$, then $\vec{OP}$ is called the *position vector* of P. If $P = (x, y, z)$, the position vector of P is denoted $\langle x, y, z\rangle$.

We will work primarily with vectors in coordinatized space. Will will denote the set of vectors in n-dimensional cartesian space by $\mathcal{R}^n$. Note that $\mathcal{R}^n$ is essentially the same set as $\mathbb{R}^n$; the difference is merely a matter of whether we are thinking of ordered n-tuples as points or as equivalence classes of arrows. In fact, unlike the case of $\mathcal{V}^n$ and $\mathcal{R}^n$, where the correspondence depends on the coordinate system chosen, there is only one natural way to create the correspondence between $\mathcal{R}^n$ and $\mathbb{R}^n$. Thus we consider these sets as "identified" and flip between these contexts without further comment.

We may denote elements of $\mathcal{V}^n$ (or $\mathcal{R}^n$) as $\vec{v}$. To each vector in coordinatized space, there exists a corresponding line through the origin and a corresponding line segment. In this case, $\bar{v}$ means the line segment corresponding to the vector $\vec{v}$. We say that two vectors $\vec{v}$ and $\vec{w}$ are *parallel* and write $\vec{v}\|\vec{w}$ if the corresponding lines are identical, and we say that $\vec{a}$ and $\vec{b}$ are *perpendicular*, or *orthogonal*, and write $\vec{a} \perp \vec{b}$ if the corresponding lines are perpendicular.

Proposition 2. *Let* $\langle x, y, z\rangle \in \mathcal{R}^3$. *Then* $|\langle x, y, z\rangle| = \sqrt{x^2 + y^2 + z^2}$.

Proof. The length of the vector is the length of the line segment from $(0, 0, 0)$ to (x, y, z). We have seen that this quantity is $\sqrt{x^2 + y^2 + z^2}$. □

11. Vector Operations

We will define four operations involving vectors. Each will be defined geometrically on vectors in affine space and algebraically on vectors in cartesian space. Initially we will put squares around the vector operations, but after we have shown that the definitions yield the same result in cartesian space, we will drop the squares.

12. Vector Addition

We define *geometric vector addition* as follows. Let $\vec{v}, \vec{w} \in \mathcal{V}^n$. Let $\vec{PQ}$ be a representative for $\vec{v}$ and select the representative for $\vec{w}$ whose tail is Q. This will be of the form $\vec{QR}$. Define $\vec{v} \boxplus \vec{w} = \vec{PR}$. This is the diagonal of a parallelogram.

The *negative* of a vector $\vec{v}$ is the vector $-\vec{v}$ such that $\vec{v} \boxplus (-\vec{v}) = \vec{0}$, the zero vector. By construction, we see that the negative of a vector represented by $\vec{PQ}$ is the vector represented by $\vec{QP}$. This is the vector of the same length in the opposite direction.

The above definition is geometric and does not rely on a coordinate system. However, when our affine space is coordinatized, we wish to have a simple method for computing the vector sum of two vectors.

With this in mind, we define the *algebraic vector sum* of two vectors $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w} = \langle w_1, w_2, w_3\rangle$ in cartesian space by

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle.$$

Proposition 3. *Let* $\vec{v}, \vec{w} \in \mathcal{R}^3$. *Then* $\vec{v} \boxplus \vec{w} = \vec{v} + \vec{w}$.

Proof. Let $\vec{v} = \langle v_1, v_2\rangle$ and $\vec{w} = \langle w_1, w_2\rangle$. Consider the parallelogram spanned by $\vec{v}$ and $\vec{w}$. The oriented diagonal of this parallelogram is the vector sum of $\vec{v}$ and $\vec{w}$. But the tip of this diagonal is $\langle v_1 + w_1, v_2 + w_2\rangle$.

The picture is this: Let $O = (0,0)$, $V = (v_1, v_2)$, $W = (w_1, w_2)$, $S = (v_1 + w_1, v_2 + w_2)$, $P = (w_1, 0)$, and $Q = (v_1 + w_1, v_2)$. Then $\triangle OPQ \cong \triangle VQS$ by angle-side-angle. $\square$

Since $\boxplus$ and $+$ yield the same results in $\mathcal{R}^n$, we will use the notation $+$ for both ideas from now on.

Suppose we want a vector that proceeds from the tip of $\vec{OP}$ to the tip of $\vec{OQ}$. This is merely the vector from P to Q. We want a vector such that when it is added to P, its tip is at Q. We notice that $\vec{OP} + \vec{PQ} = \vec{OQ}$. Thus $\vec{PQ} = \vec{OQ} - \vec{OP}$. Thus if $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, then the vector from P to Q is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Now if we add a vector $\vec{v}$ to itself, we get a vector in the same direction which is twice as long as $\vec{v}$. It is convenient to write $\vec{v} + \vec{v} = 2\vec{v}$. We now generalize this concept.

We define *multiplication of a scalar times a vector* as follows. Let $t \in \mathbb{R}$; we call t a *scalar*. Let $\vec{v} \in \mathcal{V}^n$ be a vector. Define $t\vec{v}$ to be the vector whose length is $t|\vec{v}|$. in the direction of $\vec{v}$ if $t > 0$ and in the direction of $-\vec{v}$ if $t < 0$.

Proposition 4. *Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then $t\vec{v} = \langle tv_1, tv_2, tv_3 \rangle$.*

Proof. The point (tv_1, tv_2, tv_3) is on the same line through the origin as the point (v_1, v_2, v_3). If $t > 0$, then (tv_1, tv_2, tv_3) is on the side side of the origin as the point (v_1, v_2, v_3); otherwise, it is on the opposite side. Thus the direction of $\langle tv_1, tv_2, tv_3 \rangle$ is as required. Its length is $\sqrt{t^2v_1^2 + t^2v_2^2 + t^2v_3^2} = t\sqrt{v_1^2 + v_2^2 + v_3^2}$, also as required. $\square$

Proposition 5. *Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$. The following are equivalent:*

- **i.** $\vec{v} \| \vec{w}$;
- **ii.** $\vec{v} = k\vec{w}$ *for some constant* $k \in \mathbb{R}$;
- **iii.** $\frac{v_1}{w_1} = \frac{v_2}{w_2} = \frac{v_3}{w_3} = k$ *for some constant* $k \in \mathbb{R}$.

Example 13. Let $\vec{v} = \langle 4, 4, 7 \rangle$ and $\vec{w} = \langle 2, 3, 6 \rangle$. Find a vector in the direction of $\vec{w}$ with the length of $\vec{v}$.

Solution. We have $|\vec{v}| = 9$ and $|\vec{w}| = 7$. The unit vector in the direction of $\vec{w}$ is $\frac{1}{7}\vec{w}$. Thus $\frac{9}{7}\vec{w}$ is the vector we seek. $\square$

Proposition 6. Properties of Vector Addition
Let $\vec{a}, \vec{b}, \vec{c} \in \mathcal{V}^n$ and let $d, e \in \mathbb{R}$. Then

- **(a)** $\vec{a} + \vec{b} = \vec{b} + \vec{a}$;
- **(b)** $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$;
- **(c)** $\vec{a} + \vec{0} = \vec{a}$;
- **(d)** $\vec{a} + (-1\vec{a}) = \vec{0}$;
- **(e)** $d(\vec{a} + \vec{b}) = d\vec{a} + d\vec{b}$;
- **(f)** $(d + e)\vec{a} = d\vec{a} + e\vec{a}$;
- **(g)** $(de)\vec{a} = d(e\vec{a})$;
- **(h)** $|d\vec{a}| = |d||\vec{a}|$;
- **(i)** $1\vec{a} = \vec{a}$.

Proof. Impose a coordinate system, assigning components to the vectors. Then these properties follow from properties of real numbers at the component level. For example, let $\vec{a} = \langle a_1, a_2 \rangle$. Then

$$|d\vec{a}| = |\langle da_1, da_2 \rangle| = \sqrt{d^2 a_1^2 + d^2 a_2^2} = |d|\sqrt{a_1^2 + a_2^2} = |d||\vec{a}|.$$

□

Example 14. Let $\vec{v} = \langle 4, 4, 7 \rangle$ and $\vec{w} = \langle 2, 3, 6 \rangle$. Find $|3\vec{v} - 2\vec{w}|$.

Solution. We have $3\vec{v} - 2\vec{w} = \langle 12, 12, 21 \rangle - \langle 4, 6, 12 \rangle = \langle 8, 6, 9 \rangle$. The length of this vector is $\sqrt{64 + 36 + 81} = \sqrt{181}$. □

A *linear combination* of a set of vectors $\{\vec{v_1}, \dots, \vec{v_n}\}$ is a vector of the form $a_1\vec{v_1} + \cdots + a_n\vec{v_n}$, where $a_i \in \mathbb{R}$.

The *span* of a set of vectors is the set of all linear combinations of the vectors. A pair of noncolinear vectors in $\mathbb{R}^3$ span a plane and a trio of noncoplanar vectors in $\mathbb{R}^3$ span all of space.

Let $\vec{v} \in \mathcal{V}^n$. Suppose that $\vec{w}$ and $\vec{x}$ are not on the same line. If $\vec{v} = t_1\vec{w} + t_2\vec{x}$, then t_1 is called the *component* of $\vec{v}$ in the $\vec{w}$ direction and t_2 is the component in the $\vec{x}$ direction.

A *unit vector* is a vector of length 1. If we wish to consider "pure direction" as opposed to direction and length, we use unit vectors.

To obtain a unit vector pointing in the same direction as a given vector, we merely divide by its length. Thus a unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}.$$

For each coordinate axes in $\mathbb{R}^2$ and $\mathbb{R}^3$, we give a special name to the unit vector on that axis which points in the positive direction. These are called $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively; together they are called the *standard basis vectors*. Thus for a vector $\vec{v} = \langle x, y, z \rangle$, we have that $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$. Thus x is the component of $\vec{v}$ in the $\vec{i}$ direction, and so forth.

Any point in $\mathbb{R}^3$ may be written as a linear combination of these standard basis vectors; that is, they span $\mathbb{R}^3$.

13. Projection and Dot Product

Suppose we have a flat object A in affine space such as a line or a plane, and we have another set of points X in affine space. We may *project* X onto A as follows. Since A is flat, each point in X has exactly one point in A to which it is closest. For $x \in X$, let $\mathrm{proj}_A(x)$ be that closest point. Then let $\mathrm{proj}_A(X) = \{\mathrm{proj}_A(x) \mid x \in X\}$.

Let $\vec{v} = \vec{OP}, \vec{w} = \vec{OQ} \in \mathcal{V}^n$. Let $\bar{OP}$ be the line segment from O to P and let OQ be the line through O and Q. Note that $\mathrm{proj}_{OQ}(\bar{OP})$ is a line segment. The *scalar projection* of $\vec{v}$ onto $\vec{w}$ is the signed length of the line segment $\mathrm{proj}_{OQ}(\bar{OP})$, and is denoted $\mathrm{proj}_{\vec{w}}(\vec{v})$. If $\mathrm{proj}_{\bar{OQ}}(\bar{OP}) = \{O\}$, the sign is negative; otherwise it is positive. Thus the component of $\vec{v}$ in the $\vec{w}$ direction is $\mathrm{proj}_{\vec{w}}(\vec{v})$.

The *vector projection* of $\vec{v}$ onto $\vec{w}$ is the vector of length $\mathrm{proj}_{\vec{w}}(\vec{v})$ in the same direction as $\vec{w}$; that is, $\mathrm{proj}_{\vec{w}}(\vec{v})\frac{\vec{w}}{|\vec{w}|}$.

The *geometric dot product* of two vectors $\vec{v}$ and $\vec{w}$ is defined to be

$$\vec{v} \boxdot \vec{w} = |\vec{w}|\mathrm{proj}_{\vec{w}}(\vec{v}).$$

Thus the dot product is a real number, not a vector.

Note that if u is a unit vector, then

$$\vec{v} \boxdot \vec{u} = \text{proj}_{\vec{u}}(\vec{v}).$$

For this reason, when we need to project onto vectors, we like to project onto unit vectors. In this case, the component of $\vec{v}$ in the $\vec{u}$ direction is $\vec{v} \boxdot \vec{u}$.

Now notice that for $\vec{v} = \langle x, y, z \rangle$,

- $\vec{v} \boxdot \vec{i} = \text{proj}_{\vec{i}}(\vec{v}) = x$;
- $\vec{v} \boxdot \vec{j} = \text{proj}_{\vec{j}}(\vec{v}) = y$;
- $\vec{v} \boxdot \vec{k} = \text{proj}_{\vec{k}}(\vec{v}) = z$.

If $\vec{v}$ and $\vec{w}$ are vectors, we define the *angle* between them to be the angle between corresponding line segments of representatives with the same tails. This quantity is between 0 and π radians and is denoted $\angle(\vec{v}, \vec{w})$.

Proposition 7. *Let $\vec{v}, \vec{w} \in \mathcal{V}^n$. Then*

$$\vec{v} \boxdot \vec{w} = |v||w| \cos\theta,$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

Proof. The line segment corresponding to the vector $\vec{v}$ projects onto the line segment corresponding to the vector $\vec{w}$ as the side of a right triangle with $\bar{v}$ as the hypotenuse. Thus $\vec{v} \boxdot \vec{w} = |\vec{w}|\text{proj}_{\vec{w}}(\vec{v}) = |\vec{w}||\vec{v}| \cos\theta$. □

Corollary 1. *Dot product is commutative, that is, $\vec{v} \boxdot \vec{w} = \vec{w} \boxdot \vec{v}$.*

Corollary 2. *For vectors $\vec{v}, \vec{w} \in \mathcal{V}^n$, we have*

$$|w|\text{proj}_{\vec{w}}(\vec{v}) = |v|\text{proj}_{\vec{v}}(\vec{w}).$$

Proposition 8. *Let $\vec{v}, \vec{w} \in \mathcal{V}^3$. Then $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.*

Proof. If $\theta \in [0, \pi]$, then $\cos\theta = 0$ if and only if $\theta = \pi/2$. □

We now wish to find a method of easily computing dot products. To this end, we define the *algebraic dot product* of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in coordinatized space by $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$. We will show that $\vec{v} \boxdot \vec{w} = \vec{v} \cdot \vec{w}$, giving us a convenent formula for the computation of dot product, which leads to formulas for the computation of angles and projections. First we need to state some properties of the algebraic dot product.

Proposition 9. Properties of Dot Product *Let $\vec{a}, \vec{b}, \vec{c} \in \mathcal{R}^n$ and $d \in \mathbb{R}$. Then*

- **(a)** $\vec{a} \cdot \vec{a} = |\vec{a}|^2$;
- **(b)** $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$;
- **(c)** $\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{a} \cdot \vec{b}) + (\vec{a} \cdot \vec{c})$;
- **(d)** $(d\vec{a}) \cdot \vec{b} = \vec{a} \cdot (d\vec{b}) = d(\vec{a} \cdot \vec{b})$;
- **(e)** $\vec{a} \cdot \vec{0} = 0$.

Proof. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$.

- **(a)** $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$
- **(b)** $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 = \vec{b} \cdot \vec{a}$

et cetera. □

Proposition 10. *Let $\vec{v}, \vec{w} \in \mathcal{R}^3$. Then $\vec{v} \boxdot \vec{w} = \vec{v} \cdot \vec{w}$.*

Lemma 1. Law of Cosines *For a triangle with angles A, B, C and corresponding opposite sides of length a, b, c, we have*

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

Proof. (Of Lemma for B and C acute angles - other cases similar). Drop a perpendicular from the angle A to the opposite side. Call this distance h. Let m be the distance from C to the perpendicular. Then $a - m$ is the distance from B to the perpendicular. Thus $(a-m)^2 + h^2 = c^2$ and $m^2 + h^2 = b^2$. Substituting $h^2 = b^2 - m^2$ into the first of these yields $a^2 - 2am + b^2 = c^2$. But $m = b\cos(C)$, proving the result. □

Proof. (Of proposition). The line segments corresponding to the vectors $\vec{v}$, $\vec{w}$, and $\vec{w} - \vec{v}$ form the sides of a triangle. If θ is the angle between $\vec{v}$ and $\vec{w}$, then θ is opposite the side $\vec{w} - \vec{v}$. By the Law of Cosines we have

$$|\vec{w} - \vec{v}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos(\theta).$$

Multiplying out the right side via properties (1) through (4) above yields

$$\begin{aligned} |\vec{w} - \vec{v}|^2 &= (\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v}) \\ &= (\vec{w} \cdot \vec{w}) - 2(\vec{v} \cdot \vec{w}) + (\vec{v} \cdot \vec{v}) \\ &= |\vec{w}|^2 + |\vec{v}|^2 - 2(\vec{v} \cdot \vec{w}). \end{aligned}$$

Putting this into the first equation and simplifying yields

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\theta) = \vec{v} \boxdot \vec{w}.$$

□

Thus the geometric and algebraic dot products are the same, and we no longer make any distinction between them.

Corollary 3. *If θ is the angle between the nonzero vectors $\vec{v}$ and $\vec{w}$, then*

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}.$$

Example 15. Let $\vec{v} = \langle 5, 2, 1 \rangle$ and $\vec{w} = \langle 3, 2, 3 \rangle$. Find the scalar and vector projections of $\vec{v}$ onto $\vec{w}$, and find the angle between them.

Solution. We know that $\vec{v} \cdot \vec{w} = |\vec{w}|\mathrm{proj}_{\vec{w}}(\vec{v})$. Thus

$$\mathrm{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{15 + 4 + 3}{\sqrt{9 + 4 + 9}} = \frac{22}{\sqrt{22}} = \sqrt{22}.$$

Thus the scalar projection is the length of $\vec{w}$, so vector projection is $\vec{w}$ itself. This says that $\vec{v}$ and $\vec{w}$ form a right triangle.

We also know that $\cos\angle(\vec{v}, \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{\sqrt{22}}{\sqrt{29}}$, so the angle is approximately 29.4 degrees. □

Example 16. Let $\vec{v} = \langle 5, 2, 1 \rangle$ and $\vec{w} = \langle 3, 2, 3 \rangle$. Verify that these vectors form a right triangle.

Solution. Let $\vec{x} = \vec{w} - \vec{v} = \langle -2, 0, 2 \rangle$. Then $\vec{x} \cdot \vec{w} = -6 + 0 + 6 = 0$, so $\vec{x}$ is orthogonal to $\vec{w}$. □

14. Cross Product

It will be essential for us to be able to construct a vector which is perpendicular to the plane determined by two other vectors. Towards this end, we introduce a new product which produces such a vector. This product is defined only for vectors in $\mathbb{R}^3$.

The *geometric cross product* of two vectors $\vec{v}$ and $\vec{w}$ is denoted $\vec{v} \boxtimes \vec{w}$ and if $\vec{x} = \vec{v} \boxtimes \vec{w}$, it is defined by the properties:

(1) $\vec{x} \perp \vec{v}$ and $\vec{x} \perp \vec{w}$ so that $\vec{x}$ is perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$;
(2) the length of $\vec{x}$ is equal to the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$;
(3) $\vec{x}$ is oriented by the right hand rule.

These three properties determine a unique vector.

Proposition 11. *Let $\vec{v}, \vec{w} \in \mathcal{V}^3$. Then $|\vec{v} \boxtimes \vec{w}| = |\vec{v}||\vec{w}| \sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$.*

Proof. The area of the parallelogram determined by $\vec{v}$ and $\vec{w}$ is given by area equals base times height. If we let $|\vec{v}|$ be the base, then the height is simply $|\vec{w}| \sin\theta$. □

Proposition 12. *Let $\vec{v}, \vec{w} \in \mathcal{V}^3$. Then $\vec{v} \| \vec{w}$ if and only if $\vec{v} \times \vec{w} = \vec{0}$.*

Proof. If $\theta \in [0, \pi]$, then $\sin\theta = 0$ if and only if $\theta = 0$ or $\theta = \pi$. □

Proposition 13.

(a) $\vec{i} \boxtimes \vec{j} = \vec{k}$;
(b) $\vec{j} \boxtimes \vec{k} = \vec{i}$;
(c) $\vec{k} \boxtimes \vec{i} = \vec{j}$.

As before, we will define an algebraic method for computing the cross product.

Thus we define the *algebraic cross product* of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in cartesian 3-space by

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle.$$

We remember this formula via a symbolic determinant. The determinant of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The determinant of a 3×3 matrix is

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} + a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}.$$

Thus

$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

Proposition 14. *Let $\vec{v}, \vec{w} \in \mathcal{R}^3$. Then $\vec{v} \boxtimes \vec{w} = \vec{v} \times \vec{w}$.*

Proof. Let $\vec{v} = \langle v_1, v_2, v_3\rangle$ and $\vec{w} = \langle w_1, w_2, w_3\rangle$.

(1) To see that $\vec{v} \times \vec{w} \perp \vec{v}$, we use the dot product.

$$\begin{aligned}(\vec{v} \times \vec{w}) \cdot \vec{v} &= (v_2w_3 - v_3w_2)v_1 + (v_3w_1 - v_1w_3)v_2 + (v_1w_2 - v_2w_1)v_3 \\ &= v_2w_3v_1 - v_3w_2v_1 + v_3w_1v_2 - v_1w_3v_2 + v_1w_2v_3 - v_2w_1v_3 \\ &= 0.\end{aligned}$$

Similarly, $(\vec{v} \times \vec{w}) \cdot \vec{w} = 0$ so $\vec{v} \times \vec{w} \perp \vec{w}$.

(2) We wish to show that $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$, where θ is the angle between $\vec{v}$ and $\vec{w}$.

Consider

$$\begin{aligned}|\vec{v} \times \vec{w}|^2 &= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \\ &= v_2^2w_3^2 - 2v_2v_3w_2w_3 + v_3^2w_2^2 \\ &+ v_3^2w_1^2 - 2v_1v_3w_1w_3 + v_1^2w_3^2 \\ &+ v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2.\end{aligned}$$

Also,

$$\begin{aligned}(|\vec{v}||\vec{w}| \sin(\theta))^2 &= |\vec{v}|^2|\vec{w}|^2 \sin^2(\theta) \\ &= |\vec{v}|^2|\vec{w}|^2(1 - \cos^2(\theta)) \\ &= |\vec{v}|^2|\vec{w}|^2 - |\vec{v}|^2|\vec{w}|^2 \cos^2(\theta) \\ &= |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 \\ &= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) \\ &- (v_1w_1 + v_2w_2 + v_3w_3)^2 \\ &= v_2^2w_3^2 - 2v_2v_3w_2w_3 + v_3^2w_2^2 \\ &+ v_3^2w_1^2 - 2v_1v_3w_1w_3 + v_1^2w_3^2 \\ &+ v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2.\end{aligned}$$

These last quantities are the same; taking square roots and noting that $\sqrt{\sin(\theta)} = \sin(\theta)$ since $\theta \in [0, \pi]$ yields the result.

(3) The orientation of $\vec{v} \times \vec{w}$ is actually determined by the orientation given to the coordinate axes. The proof of this requires more advanced techniques than we currently have. The basic idea is the $\vec{v} \times \vec{w}$ changes continuously as the lengths of $\vec{v}$ and $\vec{w}$ and the angle between them change. Thus if can move $\vec{v}$ to $\vec{i}$ and $\vec{w}$ to $\vec{j}$ without getting a zero vector as the cross product, the orientation of $\vec{v} \times \vec{w}$ must be the same as that of $\vec{i} \times \vec{j}$, which is right handed.

□

Example 17. Find the area of the triangle with vertices $P(2, 4, 1)$, $Q(1, 2, 3)$, and $R(5, 0, 1)$.

Solution. Treat P as a "translated origin". Let $\vec{v} = Q - P = \langle -1, -2, 2\rangle$ and $\vec{w} = R - P = \langle 3, -4, 0\rangle$. The area of the triangle is half of the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$, which we find via the cross product:

$$\vec{v} \times \vec{w} = (0 - 8)\vec{i} - (0 - 6)\vec{j} + (4 - (-6))\vec{k} = \langle -8, 6, 10\rangle.$$

Thus the area of the triangle is half to the length of this vector:

$$\text{area} = \frac{1}{2}\sqrt{64 + 36 + 100} = 5\sqrt{2}.$$

□

Proposition 15. Properties of Cross Product *Let $\vec{a}, \vec{b}, \vec{c} \in \mathcal{V}^n$ and $d \in \mathbb{R}$. Then*

(a) $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$;
(b) $(d\vec{a}) \times \vec{b} = \vec{a} \times (d\vec{b}) = d(\vec{a} \times \vec{b})$;
(c) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$;
(d) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$;
(e) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$;
(f) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$;
(g) $\vec{a} \times \vec{0} = \vec{0}$.

Proof. Write each vector in terms of their components and use the algebraic definition of cross product. □

Example 18. Let $\vec{v} = \langle 2, 5, 1\rangle$ and $\vec{w} = \langle 3, 1, 2\rangle$. Find a vector which is perpendicular to both $22\vec{v} + 29\vec{w}$ and $83\vec{v} - 8\vec{w}$.

Solution. These vectors are linear combinations of $\vec{v}$ and $\vec{w}$, and an therefore on the plane spanned by $\vec{v}$ and $\vec{w}$. It suffices to find a vector which is perpendicular to this plane. We do this by crossing $\vec{v}$ and $\vec{w}$:

$$\vec{v} \times \vec{w} = (10 - 5)\vec{\imath} - (4 - 3)\vec{\jmath} + (2 - 15)\vec{k} = \langle 5, -1, -13\rangle.$$

□

Proposition 16. *Let $\vec{a}, \vec{b}, \vec{c} \in \mathcal{V}^3$. Then $\vec{a} \cdot (\vec{b} \times \vec{c})$ is a scalar quantity which is equal to the signed volume of the parallelepiped determined by the three vectors. The magnitude of this quantity is the volume and the sign detects whether the vectors have a right or left handed orientation in the order presented. We call $\vec{a} \cdot (\vec{b} \times \vec{c})$ the* scalar triple product.

Proof. The volume is equal to the base times the height. If $\vec{x} = \vec{b} \times \vec{c}$, the height is simply the projection of $\vec{a}$ onto this vector, $\text{proj}_{\vec{x}}(\vec{a}) = \vec{a} \cdot \vec{x}/|\vec{x}|$. But the area of the base is $|\vec{x}|$, so the base times the height is $\vec{a} \cdot \vec{x}$. □

The triple scalar product can the computed as a determinant.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

Example 19. Do the points $O(0, 0, 0)$, $P(1, 2, 3)$, $Q(2, 3, 1)$, and $R(3, 1, 2)$ lie on the same plane?

Solution. We treat P, Q, and R as vectors starting at the origin, and note that the four points lie on the same plane if and only if the volume of the parallelepiped spanned by these vectors is zero. The triple scalar product is

$$P \cdot (Q \times R) = (6 - 1)1 - (4 - 3)2 + (2 - 9)3 = 5 - 2 - 21 = 18 \neq 0;$$

so no, they don't lie on the same plane. □

Example 20. Prove that the maximum volume of a parallelepiped with sides of length one is a cube.

Solution. First, draw a picture and give everything in the picture a name. Let $\vec{a}, \vec{b}, \vec{c} \in \mathcal{V}^3$. Let $\vec{x} = \vec{b} \times \vec{c}$. Let θ be the angle between $\vec{b}$ and $\vec{c}$. Let ϕ be the angle between $\vec{a}$ and $\vec{x}$. Note that $\theta, \phi \in [0, \pi]$.

The volume of a parallelepiped is base times height. The area of the base is the length of the cross product; since we have unit vectors, this is $\sin\theta$. The height is the projection of $\vec{a}$ onto $\vec{x}$; since $\vec{a}$ has unit length, this is $\cos\phi$. Thus the volume is $\sin\theta\cos\phi$.

To maximize this product, maximize each of the factors; $\sin\theta$ is largest when $\theta = \pi/2$ and $\cos\phi$ is largest when $\phi = 0$. Thus the volume is maximized when $\vec{b} \perp \vec{c}$ and $\vec{a} \| \vec{x}$, which means that $\vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$. This is a cube. □

15. Summary

- Given a "real-life" problem the geometry of the situation lies initially in affine space and we may impose the coordinate system which is most convenient. This is the reason for the distinction between affine space and cartesian space.
- Arrows have position, direction, and magnitude. The set of arrows is labeled $\mathcal{A}^n$. Vectors have only direction and magnitude. The set of vectors is labeled $\mathcal{V}^n$. Two arrows with the same magnitude and direction "represent" the same vector. We think of vectors as arrows which we can "slide around", to be placed at any convenient tail.
- The set of vectors labeled with coordinates is $\mathcal{R}^n$. There is no geometric difference between $\mathcal{V}^n$ and $\mathcal{R}^n$. The reason for the distinction is that there is more than one way to impose coordinates on $\mathcal{V}^n$; every rotation of an axis system gives a different correspondence
$$\mathcal{V}^n \leftrightarrow \mathcal{R}^n.$$
- The coordinates of a vector in $\mathcal{R}^n$ are the coordinates of its tip when its tail is at the origin. This gives a natural correspondence
$$\mathcal{R}^n \leftrightarrow \mathbb{R}^n.$$
- The operations of vector addition, scalar multiplication, dot product, and cross product are defined geometrically in $\mathcal{V}^n$ and algebraically in $\mathcal{R}^n \leftrightarrow \mathbb{R}^n$. The correspondence $\mathcal{V}^n \leftrightarrow \mathcal{R}^n$ respects these operations. Thus we may think of vectors as equivalence classes of arrows or as points, and flip between these ways of thinking at will.
- The dot product of two vectors is the length of the projection of one onto the other, adjusted by the length of the other.
- The cross product of two vectors is perpendicular to both of them, with length equal the area of the parallelogram determined by them, oriented by the right hand rule.
- Many formulas relating dot and cross products to projections, angles, and so forth can be derived from the above interpretations using pictures and simple geometric facts, and then computed with the algebraic definitions.
- The purpose of describing vectors in this way is to build up geometric intuition which will be helpful in solving problems using vector calculus.

16. Exercises

Exercise 1. Let A, B, and C be any sets. Determine which of the following statements is true, using Venn diagrams if necessary:

(a) $A \subset B \Rightarrow A \cap B = A$
(b) $A \subset B \Rightarrow B \setminus A = B$
(c) $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$
(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Exercise 2. Let $\mathbb{R}$ be the set of real numbers. For $a, b \in \mathbb{R}$, let $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ be the closed interval between a and b. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ be the set of integers. Note that $\mathbb{Z} \subset \mathbb{R}$. How many elements are contained in the following sets?

(a) $([-2, 3] \cup [5, 9]) \cap \mathbb{Z}$
(b) $([\sqrt{2}, \pi] \cup (3^3, 2^5]) \cap \mathbb{Z}$
(c) $([1, 5] \times (3, 6)) \cap (\mathbb{Z} \times \mathbb{Z})$

Exercise 3. Graph the box whose diagnonal vertices are the points $(0, 0, 0)$ and $(1, 4, 2)$. Label each vertex of the box.

Exercise 4. Let $A = [0, 1]$, $B = [1, 2)$, and $C = (3, 4]$. Graph the set $A \times A \times (B \cup C)$.

Exercise 5. Describe (and sketch if possible) the graph of the following equations:

(a) $z = 2$
(b) $(x^2 + y^2)z = 0$
(c) $x^2 + y^2 + z^2 = 0$
(d) $x^2 + y^2 + z^2 + 4 = 0$

Exercise 6. Find the center and the radius of the sphere which is the solution set of the equation

$$x^2 + y^2 + z^2 = 4x + 9y + 36z.$$

Graph the sphere.

Exercise 7. Consider the line seqment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$. Convince yourself that its midpoint is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Exercise 8. Find an equation of a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.

Exercise 9. Something to think about regarding calculators:

(a) Is the product of a rational number and an irrational number ever rational?
(b) If your answer to a problem contains $\sqrt{2}$ and/or π, does this tell you anything about the geometry of the problem?

Exercise 10.
E&P § 12.3 # 1*,5*,9,15,23,37
E&P § 13.1 # 1,3
(* = skip the perpendicular part for now)

Exercise 11. Let $\vec{v} = \langle 1, 2, 3 \rangle$ and $\vec{w} = \langle 3, 2, 1 \rangle$. Find the following, avoiding calculation where possible:

(a) $|\vec{v}|$
(b) $|\vec{w}|$
(c) $\vec{a} = \vec{v} + \vec{w}$
(d) $\vec{b} = \vec{v} - \vec{w}$
(e) $|\vec{a}|$
(f) $|\vec{b}|$
(g) $\vec{c} = 3\vec{a} + 3\vec{b}$
(h) $|\vec{c}|$

Exercise 12. Let $\vec{a} = \langle 1, 2, 3\rangle$, $\vec{b} = \langle -2, 0, -3\rangle$, and $\vec{c} = \langle 1, -2, 0\rangle$.

(a) Draw each of these vectors emanating from the origin.
(b) Now draw $\vec{a}$ eminating from the origin, $\vec{b}$ with its tail at the tip of $\vec{a}$, and $\vec{c}$ with its tail at the tip of $\vec{b}$.
(c) Find $\vec{a} + \vec{b} + \vec{c}$. Does your result agree with your picture?

Exercise 13. (*Challenge*) The spheres $x^2 + y^2 + z^2 = 144$ and $(x - 3)^2 + (y - 4)^2 + (z - 12)^2 = 25$ intersect in a circle. Find the center of the circle. (Hint: Let $P = (3, 4, 12)$, Q be a point of intersection of the spheres, and R be the center point. Use the trigonometric properties of pythagorian triples and quadruples to analyse the situation. Note that the line through Q and R is perpendicular to the line connecting the centers of the spheres.)

Exercise 14.
E&P § 12.3 # 17,19,24,31
E&P § 13.1 # 11,15,47,51,52

Exercise 15. Show that the vector

$$\vec{v} = \vec{b} - \frac{\vec{a} \cdot \vec{b}}{|a|^2}\vec{a}$$

is orthogonal to $\vec{a}$.

Exercise 16. Let $\vec{v}, \vec{w} \in \mathbb{R}^2$. Give a geometric interpretation of and prove the following formulae:

(a) Cauchy Schwarz Inequality:

$$|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$$

(b) Triangle Inequality:

$$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$$

(c) Parallelogram Law:

$$|\vec{v} + \vec{w}|^2 + |\vec{v} - \vec{w}|^2 = 2|\vec{v}|^2 + 2|\vec{w}|^2$$

(Hint for (b) and (c): Use the Cauchy Schwarz Inequality, the distributivity of dot over sum, and the fact that $|\vec{v} + \vec{w}|^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w})$.)

Exercise 17. (*Challenge*) Let f be the real valued function defined by

$$f(t) = \frac{\text{proj}_{\langle 1,1\rangle}(\langle t, t^2\rangle)}{t^2}.$$

Interpret $\lim_{t\to\infty} f(t)$ geometrically. (Hint: For several values of t, draw the vectors $\langle t, t^2\rangle$ and $\frac{\langle t,t^2\rangle}{t^2}$. Consider the limit of the slope of the function t^2 as t approaches infinity.)

Exercise 18.
E&P § 13.2 # 1,3,7,11,26

Exercise 19. Find the volume of the parallelepiped determined by the vectors $\vec{a} = \langle 1, 2, 3\rangle$, $\vec{b} = \langle 2, 3, 1\rangle$, and $\vec{c} = \langle -1, 0, z\rangle$. Find z such that these vectors are coplanar.

Exercise 20. Do the points $P(0, 1, 2)$, $Q(3, 7, 5)$, $R(-1, 0, 1)$, and $S(6, 2, 8)$ lie on the same plane? Can one change this answer by changing the y-coordinate of Q? What does this tell you?

Exercise 21. The following identities are true. Examine them for geometric content.

(a) $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$;
(b) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$;
(c) $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$.

(Hint: first consider the case of standard basis vectors; then consider the case of arbitrary unit vectors; then try to generalize to arbitrary vectors.)

Exercise 22. Let $f(t)$ be a real valued function given by

$$f(t) = |\vec{i} \times \langle \cos t, \sin t, 0\rangle|.$$

Find f and interpret it geometrically, thinking of t as time and noting that as t changes, $\langle \cos t, \sin t, 0\rangle$ sweeps out a unit circle in the xy-plane.

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