# LINES AND PLANES

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## 1. Lines

With the correspondence between points in $\mathbb{R}^n$ and vectors in $\mathcal{R}^n$ now firmly established, we now blur this distinction and jump between these contexts with impunity.

A line in $\mathbb{R}^3$ is determined by a point on the line and the direction of the line. The direction may be specified by a *direction vector*.

Let $P_0 = (x_0, y_0, z_0)$ be a given point and let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be a vector. If we start at $P_0$ and move in the direction $\vec{v}$ for a period of time $t$ at a rate given by $|\vec{v}|$, we arrive at the point

$$P = P_0 + t\vec{v}.$$

If we let $t$ range throughout the real numbers, then the set of points satisfying this equation form a line. This is called the *vector equation* of the line.

If we label $P = (x, y, z)$, then

$$(x, y, z) = (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3).$$

This gives us three equations

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3.$$

These are called the *parametric equations* of the line. The variable $t$ is called the *parameter*.

**Example 1.** Find the vector and parametric equations of the line which passes through the points $Q(1, 3, 2)$ and $R(5, -2, 3)$.

*Solution.* Let $\vec{v}$ be the vector from $Q$ to $R$. Thus $\vec{v} = R - Q = \langle 4, -5, 1 \rangle$. This is the direction of the line we seek. Letting $Q$ be a point on the line, we have that a point $P$ is on the line if $P = Q + t\vec{v} = \langle 1 + 4t, 3 - 5t, 2 + t \rangle$. Thus the parametric equations of the line become $x = 1 + 4t$, $y = 3 - 5t$, and $z = 2 + t$. □

If $v_1$, $v_2$, and $v_3$ are nonzero, we may eliminate the parameter $t$ by simply solving the parametric equations for $t$ and setting all the results equal to each other. This yields

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}.$$

These are called the *symmetric* equations of the line.

In this form, the symmetric equations point out that the locus of $P = P_0 + t\vec{v}$ is somewhat independent of $t$; we could replace $t$ by $2t$ or $t^3$ and achieve the same line. Also the symmetric equations yield the following relationships:

$$\frac{y - y_0}{x - x_0} = \frac{v_1}{v_2}; \qquad \frac{z - z_0}{x - x_0} = \frac{v_1}{v_3}; \qquad \frac{z - z_0}{y - y_0} = \frac{v_2}{v_3}.$$

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*Date*: June 28, 1995.

These are the equations may be recognized as the equations of the *projected lines*; that is, the line in $\mathbb{R}^3$ may be projected each of the coordinate planes, producing a line there whose equation is retreived from the symetric equations in this way.

**Example 2.** Find the slope-intercept form of the equation of the line which is the projection of the line $\langle 1+4t, 3-5t, 2+t\rangle$ onto the $xy$-plane.

*Solution.* We merely eliminate the third coordinate. Thus the vector equation of the line is $\langle 1+4t, 3-5t\rangle$. Eliminating $t$ yields $\frac{x-1}{4} = \frac{y-3}{-5}$. Thus $y-3 = -\frac{5}{4}(x-1)$, so $y = -\frac{5}{4}x + \frac{7}{4}$. □

Given two lines in $\mathbb{R}^3$, exactly one of the following holds:

- They intersect in exactly one point.
- Their direction vectors are parallel, in which case we call them *parallel lines*.
- They do not lie on the same plane, in which case we call them *skew lines*.

That two distinct intersecting lines on the same plane intersect in exactly one point is a result of Euclid's controversial fifth postulate. The only other claim being made here is the following intuitively clear proposition.

**Proposition 1.** *Two distinct lines have parallel direction vectors if and only if they lie on the same plane but do not intersect.*

**Example 3.** Determine whether or not the lines $\langle 2+t, 3+2t, 4+3t\rangle$ and $\langle -3+2t, 3-t, -1+t\rangle$ are parallel, intersecting, or skew.

*Solution.* The direction vectors of the lines are $\langle 1,2,3\rangle$ and $\langle 2,-1,1\rangle$, which are not parallel; thus the lines are not parallel. Two see if they intersect, let us call the parameter of the second line $s$ instead of $t$. Thus the second line becomes $\langle -3+2s, 3-s, -1+s\rangle$.

The question becomes whether or not there are real numbers $s$ and $t$ such that $2+t = -3+2s$, $3+2t = 3-s$, and $4+3t = -1+s$. We assume that there is such an $s$ and $t$ and try to find them. Adding the last two equations gives $7+5t = 2$ so $5t = -5$ and $t = -1$. If $t = -1$, then the last equation gives $4-3 = -1+s$ so $s = 2$. Now plug $t = -1$ and $s = 2$ into our lines and see that they give the same point, $(1,1,1)$. Thus the lines intersect there. □

## 2. Planes

A plane in $\mathbb{R}^3$ is determined by a point on the plane and a perpendicular direction. A vector which is perpendicular to a plane is called a *normal vector*.

Let $P_0 = (x_0, y_0, z_0)$ be a given point and let $\vec{n} = \langle n_1, n_2, n_3\rangle$ be a vector. Suppose $P = (x,y,z)$ is a point on the plane which passes through $P_0$ and is perpendicular to $\vec{n}$. Then the arrow from $P_0$ to $P$ is on the plane, and the vector $P - P_0$ is perpendicular to the normal vector $\vec{n}$. Thus

$$(P - P_0)\cdot\vec{n} = 0.$$

The set of all $P$ which satisfy this equation constitute the plane; this is called the *vector equation* of the plane.

Writing this in coordinates gives $(x-x_0, y-y_0, z-z_0)\cdot(n_1,n_2,n_3) = 0$ so

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0,$$

which can also be written

$$n_1 x + n_2 y + n_3 z = n_1 x_0 + n_2 y_0 + n_3 z_0.$$

On the other hand, the locus of an equation

$$ax + by + cz = d$$

is a plane with normal vector $\langle a, b, c \rangle$. This is called the *standard form* of the equation of a plane.

**Example 4.** Find the equation of the plane which passes through the point $P(2, 4, 1)$ with normal vector equal to the position vector.

*Solution.* We have that $\vec{n} = \langle 2, 4, 1 \rangle$. The equation of the plane, then, is $2(x - 2) + 4(y - 4) + (z - 1) = 0$, which simplifies to $2x + 4y + z = 21$. $\square$

If the plane is presented in the standard form $ax + by + cz = d$ and $a, b, c, d$ are positive, the plane is particularly easy to graph. Simply find the axis intercepts by setting two of the variables to zero.

$$x - \text{intercept} = \frac{d}{a} \qquad y - \text{intercept} = \frac{d}{b} \qquad z = \text{intercept} = \frac{d}{c}$$

Plot these points and connect the dots to obtain a nice picture of the plane.

**Example 5.** Find the equation of the plane which passes through the points $P(3, 0, 0)$, $Q(0, 2, 0)$, and $R(0, 0, 5)$.

*Solution.* The plane is of the form $ax + by + cz = d$. Let $d = 3 \cdot 2 \cdot 5 = 30$. We know that $3 = \frac{d}{a} = \frac{30}{a}$, so $a = 10$. Similarly, $b = 15$ and $c = 6$. Thus our plane is $10x + 15y + 6z = 30$. $\square$

**Example 6.** Find the equation of the plane which passes through the points $P(2, 1, 3)$, $Q(1, 5, 3)$, and $R(3, 2, 5)$.

*Solution.* The vectors $\vec{v} = Q - P = \langle -1, 4, 0\rangle$ and $\vec{w} = R - P = \langle 1, 1, 2\rangle$ lie on the plane. Thus their cross product is perpendicular to it, so we may use this as a normal vector.

$$\vec{n} = \vec{v} \times \vec{w} = \langle 6, 2, -5\rangle$$

Then use $P$ as a point on the plane, which gives the equation $6(x - 2) + 2(y - 1) - 5(z - 3) = 0$, which simplifies to $6x + 2y - 5z = 6$. □

Given two planes, exactly one of the following holds:

- Their normal vectors are parallel, in which case they are said to be *parallel planes*.
- They intersect in a line.

The *angle* between two planes is the angle between their normal vectors.

**Example 7.** Let $\vec{v} = \langle 1, 2, 2\rangle$ and $\vec{w} = \langle 2, 0, 1\rangle$. Let $Y$ be the plane spanned by $\vec{v}$ and $\vec{j}$ and let $Z$ be the plane spanned by $\vec{w}$ and $\vec{k}$. Find the line which is $Y \cap Z$ and the angle between $Y$ and $Z$.

*Solution.* Outline:

(1) Find the normal vectors using cross product;
(2) Cross the normals to find the direction vector of the line;
(3) Find a point on the line to produce the equation of the line;
(4) Dot the normals to find the angle between them.

□

**Example 8.** Let $T$ be the plane given by $5x + 3y + z = 4$ and let $P = (6, 2, 7)$. Find the distance from $P$ to $T$.

*Solution Method 1.* Find the line through $P$ in the direction of the normal vector of the plane. This line intersects the plane at a point $Q$. Then find the distance between $P$ and $Q$. □

*Solution Method 2.* Find any point $Q$ on the plane. Let $\vec{v} = P - Q$. Find the unit normal $\vec{n}$ to the plane. Project $\vec{v}$ onto $\vec{n}$.

We find $Q$ by plugging in arbitrary $x$ and $y$ and solving for $z$. It is easiest to use $x = 0$ and $y = 0$, which gives that $Q = (0, 0, 4)$ is on the plane.

Now find the unit normal vector of the plane. A normal vector is $\langle 5, 3, 1\rangle$, so the unit normal is $\vec{n} = \frac{\langle 5,2,1\rangle}{\sqrt{35}}$.

Project the vector $\vec{v} = P - Q = \langle 6, 2, 3\rangle$ onto the unit normal. This will give the distance.

$$\text{proj}_{\vec{n}}(\vec{v}) = \vec{n} \cdot \vec{v} = \frac{30 + 4 + 3}{\sqrt{35}} = \frac{37}{\sqrt{35}}.$$

□

## 3. Exercises

**Exercise 1.**
E&P § 13.3 # 1,5,7,11,15,17,23,31,33

**Exercise 2.** Find an equation for the plane consisting of all points that are equidistant from the two points $(1, 1, 0)$ and $(0, 1, 1)$.

**Exercise 3.** Let $A$ be the plane given by $x+2y+3z = 6$ and $B$ be the plane given by $3x + 2y + z = 6$.
Let $L = A \cap B$ be the line of intersection of $A$ and $B$. Let $P = (1, 1, 1)$ and note that $P \in L$.
Find the equation of the plane which is perpendicular to $L$ and passes through the point $P$, expressed in the form $ax + by + cz = d$.

**Exercise 4.** (*Challenge*) The volume of a tetrahedron is $V = \frac{1}{3}Ah$, where $A$ is the area of the triangular base and $h$ is the height. Let $P = (1, 2, 1)$. Some planes through $P$, together with the coordinate planes, determine a tetrahedron.

Let $Q = (4, 3, 4)$ and let $\vec{v} = \langle -2, 1, 0\rangle$. Suppose there is a moving plane at $P$ whose normal vector $\vec{n}$ at time $t$ is a vector from $P$ to the point $Q + t\vec{v}$. Find the minimum volume of a tetrahedron bounded by the coordinate planes and the plane through $P$ with normal vector $\vec{n} = Q + t\vec{v} - P$. (Hint: find $V$ as a function of $t$ by considering the axis intercepts of the plane determined by $\vec{n}$ for a fixed but arbitrary $t$.)

**Exercise 5.** (*Challenge*) Let $S$ be the solution set of the equation $x^2 + y^2 + z^2 = 4$ Let $P = (0, 0, 1)$. Find a vector $\vec{n}$ such that the plane through $P$ with normal vector $\vec{n}$ intersects $S$ in a circle of radius 1.

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