CURVES AND SURFACES

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1. Functions

Let A and B be sets. A *function* from A to B, denoted $f : A \to B$, is an assignment of every element in A to an element in B. We say that f *maps* A *into* B, and that f is a function *on* A. If $a \in A$, the element to which it is assigned is denoted $f(a)$. We say that a is *mapped to* b by f. We often think of a function as taking the set A and placing in in the set B, possibly compressing it in the process.

If A is sufficiently small, we may explicitly describe the function by listing the elements of A and where they go; for example, if $A = \{1, 2, 3\}$ and $B = \mathbb{R}$, a perfectly good function is described by $\{1 \mapsto 23.432, 2 \mapsto \pi, 3 \mapsto \sqrt{593}\}$.

However, if A is large, the functions which are easiest to understand are those which are specified by some *rule* or *algorithm*. The common functions of single variable calculus are of this nature, for example, the polynomials in x, $\sin x$, $\log x$, etc.

If $f : A \to B$, the set A is called the *domain* of the function and the set B is called the *codomain*.

If $C \subset A$, we define the *image* of C under f to be the set

$$f[C] = \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

The image of the domain is called the *range* of the function.

Notice that for some functions, not every point in the codomain is necessarily in the range. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, there is no point $x \in \mathbb{R}$ which is mapped to -1 by f. A function $f : A \to B$ is called *surjective* (or *onto*) if

$$\forall b \in B \exists a \in A \ni f(a) = b.$$

Equivalently, f is surjective if $f[A] = B$.

If $D \subset B$, we define the *preimage* of D under f to be the set

$$f^{-1}[D] = \{a \in A \mid f(a) \in D\}.$$

If D is a singleton set, that is if $D = \{b\}$ for some $b \in B$, we may write $f^{-1}[b]$ instead of $f^{-1}[\{b\}]$.

Notice that $f^{-1}[b]$ is not necessarily a single element in A. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, then the preimage of the point 4 is

$$f^{-1}[4] = \{2, -2\}.$$

A function $f : A \to B$ is called *injective* (or *one-to-one*) if

$$\forall a, b \in A, f(a) = f(b) \Rightarrow a = b.$$

Equivalently, f is injective if for all $b \in B$, $f^{-1}[b]$ contains at most one element in A.

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A function $f : A \to B$ is called *bijective* if it is both injective and surjective. Such a function sets up a *correspondence* between the elements of A and the elements of B.

Example 1. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto 2n$ is injective but not surjective. The function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ given by $(p, q) \mapsto \frac{p}{q}$ is surjective but not injective. $\square$

Example 2. Let $\mathcal{A}^3$ be the set of arrows in affine space. If we assign to each arrow the vector which it represents, we get a function

$$\nu : \mathcal{A}^3 \to \mathcal{V}^3.$$

Selecting a particular coordinate system for affine space gives us a function

$$\phi : \mathcal{V}^3 \to \mathcal{R}^3.$$

We get one such ϕ for every selection of axes emanating from the point we select as the origin. Since we have identified $\mathcal{R}^3$ with $\mathbb{R}^3$, we may think of ϕ as a function

$$\phi : \mathcal{V}^3 \to \mathbb{R}^3.$$

Thus ϕ assigns a point in $\mathbb{R}^3$ to each equivalence class of directed line segments in affine 3-space. This function is bijective, in that for every vector, we get exactly one point in $\mathbb{R}^3$, and every point in $\mathbb{R}^3$ corresponds to a vector. Note that ϕ respects the operations of vector sum, dot, and cross product. That is, $\phi(\vec{v} \oplus \vec{w}) = \phi(\vec{v}) + \phi(\vec{w})$, and so forth. $\square$

The *graph* of a function $f : A \to B$ is defined by

$$\{(a, b) \in A \times B \mid b = f(a)\}.$$

Note that we identify $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$. Thus the graph of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\{(x_1, \ldots, x_m, y_1, \ldots, y_n) \mid f(x_1, \ldots, x_m) = (y_1, \ldots, y_n)\}.$$

For example, the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is a subset of $\mathbb{R}^2$ and the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is a subset of $\mathbb{R}^3$.

2. Continuous Functions

First we describe the concepts of limit and continuity loosely for functions in general, and then specifically for the sets we are interested in.

Suppose that a set is endowed with a notion of when two points are "near" each other. Such a set is called a *topological space*. If the notion of "nearness" is an actual distance, so that for any two points there is a positive real number which is the distance between them, then the set is called a *metric space*. Two points which are "near" each other are said to be in the same *neighborhood*.

Loosely speaking, a function between such sets is said to be *continuous* if it preserves the property of "nearness"; that is, the function maps nearby points to nearby points.

More specifically, suppose that A and B are sets for which a notion of distance is defined. Suppose $f : A \to B$. Let $a_0 \in A$ so that $b_0 = f(a_0) \in B$. Now if ϵ is a small positive real number, the set of all the points whose distance to b_0 is less than ϵ is a neighborhood of b_0.

We say that f is continuous if for every such ϵ, there is a positive real number δ such that the neighborhood of radius δ around a_0 is mapped into the neighborhood of radius ϵ around b_0. This is the same as saying that for every a which is within δ of a_0, then $f(a)$ is within ϵ of b_0.

We are dealing to the sets $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, each of which carries a natural notion of distance; thus we may speak of continuous functions between these sets or subsets of these sets.

The limit concept is fundamentally tied to the concept of continuity. Let A and B be sets which carry a notion of distance, and suppose we have a function $f : A \to B$. If $a_0 \in A$, by "approaching" a_0 we mean we are considering other points $a \in A$ which are getting closer to a_0 under our notion of distance. Thus the function f has a *limit* $L \in B$ as a approaches a_0 if $f(a)$ gets closer and closer to L as a gets closer and closer to a_0. How close? As close as we want, via the epsilon and delta definition. Thus f is continuous at a point where the limit exists if the value of f at that point is the limit.

3. Vector Valued Functions

A *vector valued function* is a function whose range is $\mathcal{V}^n$. We may coordinatize affine space and obtain a function into $\mathcal{R}^n$ (which is identified with $\mathbb{R}^n$). Now we may consider the values either as equivalence classes of directed line segments or as points in cartesian space, whichever is convenient.

Thus suppose that $D \subset \mathbb{R}^m$ and let $f : \mathbb{R}^m \to \mathbb{R}^n$. Let $a_0 \in D$.

We say that the *limit* of the function f as a approaches a_0 is $L \in \mathbb{R}^n$ and write $\lim_{a \to a_0} f(a) = L$ if for every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that if $0 < |a - a_0| < \delta$ then $|f(a) - L| < \epsilon$.

We say that f is *continuous* at $a_0 \in D$ if for every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that if $|a - a_0| < \delta$ then $|f(a) - f(a_0)| < \epsilon$.

Equivalently, f is continuous at a_0 if the $\lim_{a \to a_0}$ exists and is equal to $f(a_0)$. Thus a continuous function "commutes" with the limit operator:

$$\lim_{a \to a_0} f(a) = f(\lim_{a \to a_0} a) = f(a_0).$$

Now let I be a "connected" subset of $\mathbb{R}$, i.e., I is an interval. Intuitively, a continuous function $f : I \to \mathbb{R}^3$ behaves as follows. Think of I as a thin piece of straight wire. Then f may bend, stretch, or compress the wire, but since it is continuous, it cannot break the wire. Then f places the stretched and bent wire into space.

If D is a "connected" subset of $\mathbb{R}^2$, e.g., a rectangle, and $f : D \to \mathbb{R}^3$, then think of D as a piece of sheet metal and f as a machine which bends and stretches it and places it somewhere in space.

4. Curves and Surfaces

Lines and planes in $\mathbb{R}^n$ are "flat" subsets of $\mathbb{R}^n$ of dimension 1 and 2, respectively. Curves and surfaces are subsets of $\mathbb{R}^n$ of dimension 1 and 2, respectively, which are not necessarily flat. We will study curves and surfaces which are the images, preimages, and graphs of continuous functions $f : \mathbb{R}^m \to \mathbb{R}^n$ where $m, n = 1, 2, 3$.

Note that the equation $f(P) = g(P)$ may be written $f(P) - g(P) = 0$, where $P \in \mathbb{R}^m$. Thus the locus of equation $f(P) = g(P)$ is the preimage of the point 0 under the function $f(P) - g(P)$. In general, the preimage of a point is called a *level set*.

	Curves	Surfaces
Graphs	$\{(x, f(x))\}$ where $f : \mathbb{R} \to \mathbb{R}$	$\{(x, y, f(x, y))\}$ where $f : \mathbb{R}^2 \to \mathbb{R}$
Preimages (Level Sets)	Locus of $f(x, y) = k$ where k is constant	Locus of $f(x, y, z) = k$ where k is constant
Images (Parametrizations)	Image of $f : \mathbb{R} \to \mathbb{R}^n$	Image of $f : \mathbb{R}^2 \to \mathbb{R}^n$

5. Parametrized Curves

A *path* in $\mathbb{R}^n$ is a continuous function

$$\vec{r} : I \to \mathbb{R}^n.$$

The set of points which is the image of a path is called a *curve.* The function $\vec{r}$ is called a *parametrization* of the curve. This means that $\vec{r}$ supplies a specific rule which tells us how to "trace out" the curve. There is more than one way to do this.

If we use the letter t to represent arbitrary values in I, we say that t is the *variable* of the function $\vec{r}$, or the *parameter* of the curve. Thus an arbitrary point on the curve may be written $\vec{r}(t)$ where $t \in I$. It is often convenient to think of t as time, so as t increases, time passes and $\vec{r}(t)$ moves along the curve. For this reason, we call $\vec{r}(t)$ the *position vector* of the path at time t.

Example 3. Let $P_0 \in \mathbb{R}^3$ and let $\vec{v} \in \mathcal{R}^3$. The line $P = P_0 + t\vec{v}$ is a parametrized curve. Thus if $P_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the line is parametrized by $\vec{r}(t) = \langle x_0 + v_1 t, y_0 + v_2 t, z_0 + v_3 t \rangle$. $\square$

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, e.g., $f(x) = x^2$. Let $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ be given by $\vec{r}(t) = \langle t, f(t) \rangle$. Then $\vec{r}$ traces out the graph of the function f. $\square$

Example 5. Let $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ be given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$. Then $\vec{r}$ traces out the unit circle in the plane. $\square$

Example 6. Let $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ be given by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. Then $\vec{r}$ traces out a helix which spirals around the z-axis. $\square$

Suppose that $\vec{r}$ is a path and consider the function given by looking at just one coordinate of the image of $\vec{r}$. We have one such function for every dimension in the image. These are called *coordinate functions.*

Example 7. The coordinate functions of the helix above are

$$x(t) = \cos t; \qquad y(t) = \sin t; \qquad z(t) = t.$$

$\square$

Proposition 1. *Let $f : I \to \mathbb{R}^3$ and let $t_0 \in I$. Let $f(t) = \langle x(t), y(t), z(t) \rangle$. Then $\lim_{t\to t_0}$ exists if and only if $\lim_{t\to t_0} x(t)$, $\lim_{t\to t_0} y(t)$, and $\lim_{t\to t_0} z(t)$ all exist. In this case,*

$$\lim_{t\to t_0} f(t) = \langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t) \rangle.$$

Proof. Exercise (idea: if two points are close, so are their coordinates). $\square$

Corollary 1. *A function $f : I \to \mathbb{R}^3$ is continuous at $t_0 \in I$ if and only if each of the coordinate functions are continuous at t_0.*

Now let $\vec{r} : I \to \mathbb{R}^n$. We say that $\vec{r}$ is *differentiable* at the point $t \in I$ if the limit

$$\lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

exists. If so, this limit is called the *derivative* of $\vec{r}$ at t and is denoted $\vec{r}'(t)$ or $\frac{d\vec{r}}{dt}$.

Let $\vec{w}(t) = \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$. Geometrically, we see that the line corresponding to the vector $\vec{w}(t)$ gets closer and closer to being a tangent line to the curve as $h \to 0$. At the limit, it becomes a tangent line; for this reason, we call $\vec{r}'(t)$ a *tangent vector.*

Proposition 2. *If $\vec{r}$ is a path, then $\vec{r}$ is differentiable at $t \in I$ if and only if each of the coordinate functions is differentiable at t; in this case*

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Proof. The limits of the coordinates are the coordinates of the limit. □

6. Operations on Paths

Just as we can add or multiply two real valued functions to obtain another real valued function, we can perform vector operations on paths to obtain new functions. We must be careful here, however, about the codomain of such a function.

Let $\vec{p} : I \to \mathbb{R}^3$ and $\vec{q} : I \to \mathbb{R}^3$ be two vector valued functions and let $f : I \to \mathbb{R}$ be a real valued function.

Then for a given time t,

- $\vec{p}(t) + \vec{q}(t) \in \mathbb{R}^3$;
- $f(t)\vec{p}(t) \in \mathbb{R}^3$;
- $\vec{p}(t) \cdot \vec{q}(t) \in \mathbb{R}$;
- $\vec{p}(t) \times \vec{q}(t) \in \mathbb{R}^3$.

Proposition 3. Properties of Path Differentiation *Let $\vec{p} : I \to \mathbb{R}^3$ and $\vec{q} : I \to \mathbb{R}^3$ be two vector valued functions, let $f : I \to \mathbb{R}$ be a real valued function, and let $c \in \mathbb{R}$ be a constant. Then*

(1) $\frac{d}{dt}[\vec{p}(t) + \vec{q}(t)] = \frac{d\vec{p}}{dt} + \frac{d\vec{q}}{dt}$;
(2) $\frac{d}{dt}[c\vec{p}(t)] = c\frac{d\vec{p}}{dt}$;
(3) $\frac{d}{dt}[f(t)\vec{p}(t)] = \frac{df}{dt}\vec{p}(t) + f(t)\frac{d\vec{p}}{dt}$;
(4) $\frac{d}{dt}[\vec{p}(t) \cdot \vec{q}(t)] = \frac{d\vec{p}}{dt} \cdot \vec{q} + \vec{p} \cdot \frac{d\vec{q}}{dt}$;
(5) $\frac{d}{dt}[\vec{p}(t) \times \vec{q}(t)] = \frac{d\vec{p}}{dt} \times \vec{q} + \vec{p} \times \frac{d\vec{q}}{dt}$.

Proof. Expand into coordinates and compute. □

The last three are forms of the product rule. Be careful with the last one. The order of the factors must be preserved because the cross product is not commutative.

Example 8. Let $\vec{r} : I \to \mathbb{R}^n$ be a path of constant magnitude, that is, $|\vec{r}|(t) = K$ for every $t \in I$ and some fixed constant $K \in \mathbb{R}$. Show that the tangent vector is perpendicular to the position vector. Interpret this geometrically.

Solution. We have that

$$K^2 = |\vec{r}|^2 = \vec{r} \cdot \vec{r}.$$

Taking the derivative of both sides gives

$$0 = \vec{r}' \cdot \vec{r} + \vec{r} \cdot \vec{r}' = 2\vec{r}' \cdot \vec{r}.$$

Thus $\vec{r}' \cdot \vec{r} = 0$, so $\vec{r}' \perp \vec{r}$.

Geometrically, that $\vec{r}$ has constant magnitude K tells us that $\vec{r}$ lies on a sphere of radius K. Then the position vector is on a radial line and the tangent vector is on a tangent line. Any line tangent to a sphere is perpendicular to a radial line. $\square$

7. Particular Motion

Let $\vec{r} : [a, b] \to \mathbb{R}^3$. Consider $\vec{r}(t)$ to be the position vector of a particle at time t. Then the curve which is the image of the function $\vec{r}$ describes the spatial motion of the particle. However, the curve alone is insufficient to know the rate at which the particle traces out the curve.

The *velocity vector* of the particle is the vector whose direction points in the instantaneous direction of the particle and whose length is the speed of the particle. We have already seen that the derivative $\vec{r}'(t)$ is tangent to the curve, and so indicates the direction of the particle. The speed of the particle is the rate of change in distance traveled; that is, it is the derivative of the *arclength function*, which gives the length of the curve from the starting point a to the point $t \in [a, b]$; this function is traditionally called $s(t)$. Let us compute this function.

Break up the domain $[a, t]$ into n portions of equal size $\Delta x = \frac{t-a}{n}$. Let $x_i = a + i\Delta x$ for $i = 0, \ldots, n$. Note that $x_{i+1} = x_i + \Delta x$. The length of the arc is approximated by adding together the lengths of the line segments from $\vec{r}(x_i)$ to $\vec{r}(x_{i+1})$. One such line segment corresponds to the vector

$$\vec{r}(x_i + \Delta x) - \vec{r}(x_i);$$

therefore

$$s(t) \doteq \sum_{i=0}^{n-1} \frac{|\vec{r}(x_i + \Delta x) - \vec{r}(x_i)|}{\Delta x} \Delta x.$$

The approximation becomes exact as $\Delta x \to 0$, so

$$s(t) = \int_a^t |\vec{r}'(x)| dx.$$

Thus by the Fundamental Theorem of Calculus, the rate of change of the distance function is $s'(t) = |\vec{r}'(t)|$.

Let us call the derivative of the position vector the *path tangent vector*. Since the direction of the path tangent vector is the direction of the curve, and the length of the path tangent vector is the speed of the parametrization, we see that the path tangent vector is exactly the velocity vector.

Similarly, the *acceleration vector* is the derivative of the velocity vector.

Example 9. Let $\vec{r} = \langle \cos t, \sin t, t \rangle$ be the position function of a particle traveling in a helical path. Find the velocity, acceleration, and speed of the particle at any time t. Find the arclength function. Find the vertical distance between two loops of the helix. Find the arclength of one loop.

Solution. The velocity is $\vec{v}(t) = \vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$; the acceleration is $\vec{a}(t) = \vec{v}'(t) = \langle -\cos t, -\sin t, 0 \rangle$.

The speed is the derivative of the arclength function, which is also the magnitude of the velocity vector $s'(t) = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. Thus $s(t) = \int \sqrt{2} = \sqrt{2}t$.

The helix makes one loop as t changes by 2π. Thus the vertical change is 2π and the arclength is $2\pi\sqrt{2}$. $\square$

8. Curve Analysis

When we are interested in the curve itself as opposed to its parametrization, then the direction of the velocity vector remains important but its length does not. Thus we study the *unit tangent vector*, which is any tangent vector divided by its length. In order to find the unit tangent vector from a given parametrization, we must be assured that the derivative is always defined and never zero.

A path $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ is called *smooth* if it is infinitely differentiable; that is, if the derivative is differentiable, and the second derivative is differentiable, and so forth ad infinitum.

A *regular path* is a path which is everywhere smooth and whose first derivative is never zero. Thus if $\vec{r} : I \to \mathbb{R}^n$ is regular, there is a well-defined unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}.$$

9. Path Integration

The integral of a path $\vec{r} : [a, b] \to \mathbb{R}^3$ is defined to be

$$\int_a^b \vec{r}(t)dt = \lim_{\Delta t \to 0} \sum_{i=1}^n \vec{r}(t_i^*)\Delta t,$$

where t_i^* is a point in the i^{th} subinterval of a division of $[a, b]$ into n subintervals, each with length $\Delta t = (b - a)/n$.

We may write $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, and since limit passes to the components of the function, we arrive at

$$\int_a^b \vec{r}(t)dt = \Big\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\Big\rangle;$$

that is, a vector valued function may be integrated componentwise. It is clear, then, that differentiation and integration are inverse operations on vector valued functions, since each merely acts on the components.

This may be used to retreive the velocity and position vectors from the acceleration vector. Suppose for simplicity that the initial position and velocity of a particle is the zero vector. Then since $\vec{a}(t) = \vec{v}'(t)$ and $\vec{v}(t) = \vec{r}'(t)$, we have that $\vec{v} = \int \vec{a}$ and $\vec{r} = \int \vec{v}$.

Example 10. In baseball, a pitcher's mound is 60 feet from home plate. The pitcher throws the ball directly towards the plane of the strike zone with initial speed 80 miles per hour and initial release point 2 feet above the center of the strike zone, but with spin such that the lateral component of acceleration is 16 feet per second per second towards the pitchers left. Find the position of the ball relative to the center of the strike zone as it passes the batter, ignoring wind resistence other than the lateral acceleration and assuming that the height of the release point of the ball is the same as the height of the center of the strike zone.

Solution. First note that 80 miles per hour is about 1.5 feet per second, so the initial speed is 120 feet per second. Thus it takes the ball approximately one half of a second to reach the plane of the strike zone.

Impose a coordinate system on the situation. Say the initial release point of the ball is $-60\vec{i}$ and the center of the strike zone is the origin.

Then the initial position vector is $\vec{r}_0 = \langle -60, 0, 2\rangle$ and the initial velocity vector is $\vec{v}_0 = \langle 120, 0, 0\rangle$.

The acceleration vector is constant under our assumptions. Its vertical component is caused by gravity, which is -32 feet per second squared. Thus $\vec{a}(t) = \langle 0, 16, -32\rangle$.

We find $\vec{v}(t)$:

$$\vec{v}(t) = \int \vec{a}(t)dt = \langle 0, 16t, -32t\rangle + \vec{v}_0 = \langle 120, 16t, -32t\rangle.$$

Now we find $\vec{r}(t)$:

$$\vec{r}(t) = \int \vec{v}(t)dt = \langle 120t, 8t^2, -16t^2\rangle + \vec{r}_0 = \langle 120t - 60, 8t^2, -16t^2 + 2\rangle.$$

Thus $\vec{r}(\frac{1}{2}) = \langle 0, 2, -2\rangle$. The pitch is low and away to a right handed hitter. $\square$

10. Conic Sections

Just as curves are warped "one dimensional sets" in $\mathbb{R}^2$ or $\mathbb{R}^3$, surfaces are warped "two dimensional sets" in $\mathbb{R}^3$. We begin by examining surfaces which arise as loci of equations.

Consider the equation $z^2 = x^2 + y^2$. The locus of this equation is a surface. We wish to graph this surface.

If we pass a plane through a surface, we obtain a curve. Thus to see what a surface looks like, it is convenient to fix one of the variables and see what kind of curve the intersection of this plane with the surface is.

First we set $z = 0$. We discover that the intersection of the surface with the xy-plane is a point, since $x^2 + y^2 = 0$ is satisfied only by the origin.

When $x = 0$, $z^2 = y^2$ so $z = \pm y$. This is a pair of diagonal lines in the yz-plane. Similarly, we get diagonal lines in the xz-plane.

Now let $z = k$, where $k \neq 0$. These are the horizontal slices. Here, $x^2 + y^2 = k^2$. Then the intersection of our surface with the plane at height k is a circle of radius k. We can now see that this surface is a cone.

Letting $x = k$, we obtain the vertical slices of the cone parallel to the yz-plane. These are given by the equation $z^2 = k^2 + y^2$, or $z^2 - y^2 = k^2$. These are hyperbolas. Similarly, the slices parallel to the xz-plane are hyperbolas.

Finally, consider the planes given by $y + z = k$. This plane is parallel to the x-axis and makes a 45 degree angle with the xy-plane. The intersection with the cone is $(y + k)^2 = x^2 + y^2$, so $2ky - k^2 = x^2$ and $y = \frac{x^2 - k^2}{2k} = \frac{x^2}{2k} - \frac{k}{2}$. These slices are parabolas.

For this reason, the ellipses, hyperbolas, and parabolas are called *conic sections.*

11. Quadric Surfaces

Just as curves which are the loci of degree 1 equations are always lines, surfaces which are the loci of equations of degree 1 equations are always planes.

Surfaces which are the loci of degree 2 equations of the form

$$Ax^2 + By^2 + Cz^2 + Hxy + Ixz + Jyz + Dx + Ey + Fz + K = 0$$

are called *quadric surfaces.* Quadric surfaces are the 3rd dimensional analog of conic sections. We wish to analyse these surfaces in a more or less "standard position" so that we can understand their shapes and sizes without regard to their locations (for now). Thus we may rotate, reflect, or translate the surface to simplify the equation.

By rotation and translation (in a manner we will not discuss), each quadric surface can be represented by an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + K = 0.$$

Cylinders (Curtains)

When one of the variables is missing, that variable is free to roam. Then we have a *curtain* whose horizontal slices are conic sections. To graph such a surface, merely graph the curve in the appropriate coordinate plane and extend perpendicularly. These are also called *cylinders*, although (to me) this connotes a right elliptical cylinder.

Now we assume that none of the variables is missing. Further rotations, reflections, rotations and algebraic manipulations leave us with the following cases.

(1) $Ax^2 + By^2 + Cz^2 = K$;
(2) $Ax^2 + By^2 + K = z$.

The basic shape of these surfaces is determined by the signs of A, B, and C, and whether K is nonzero. The proportions are determined by the absolute values of A, B and C. First we look at the basic shapes.

First Equation
Sphere

When all the signs are positive, we have $x^2 + y^2 + z^2 = 1$. We know this to be the unit *sphere*. All planar slices are circles.

One Sheeted Hyperboloid

When one of the signs is negative, e.g., $x^2 + y^2 - z^2 = 1$, we have a *one sheeted hyperboloid*. The axis of symmetry is the variable which has the negative sign. The horizontal slices are circles and the vertical slices are hyperbolas not intersecting the axis of symmetry.

Two Sheeted Hyperboloid

When two of the signs are negative, e.g., $-x^2 - y^2 + z^2 = 1$, we have a *two sheeted hyperboloid*. The axis of symmetry is the variable which has the positive sign. The horizontal slices are circles and the vertical slices are hyperbolas not intersecting the coordinate plane perpendicular to the axis of symmetry.

Note that the locus of $-x^2 - y^2 - z^2 = 1$ is the empty set.

Cone

We have seen that the locus of $x^2 + y^2 - z^2 = 0$ is a *cone*.

Second Equation
Elliptic Paraboloid

When both squared terms have the same sign, e.g., $z = x^2 + y^2$, we have an *elliptic paraboloid*. The horizontal slices are circles and the vertical slices are parabolas.

Hyperbolic Paraboloid (Saddles)

When the squared terms have different signs, e.g., $z = y^2 - x^2$, we have a *hyperbolic paraboloid*, or *saddle* . The horizontal slices are hyperbolas. The vertical slices in one direction are upward parabolas and the vertical slices in the other direction are downward parabolas.

If x or y is the nonsquared term, we simply have the same surface rotated to that axis.

Effects of Changing K

For $x^2 + y^2 + z^2 = K$, we know the K is the radius of the sphere.

For $x^2 + y^2 - z^2 = K$, the shape is determined by the sign of K and the size is determined by the absolute value of K.

For $z = Ax^2 + Bx^2 + K$, $(A, B = \pm 1)$, K determines the z intercept of the surface. Changing K does not change the shape, but merely translates the surface.

Effects of Changing A, B, C

In both equations, increasing the absolute value of A, B or C squashes the surface in the direction to which these coefficients attach, whereas decreasing them stretches it in that direction.

Let $a = \frac{1}{\sqrt{|A|}}$, $b = \frac{1}{\sqrt{|B|}}$, and $c = \frac{1}{\sqrt{|C|}}$. Now increasing a, b, or c stretches the surface, and considering these constants helps us graph the surface. The circular slices in the descriptions above become ellipses upon stretching.

Ellipsoids

Consider $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. What happens to the locus of this equation as a, b, c change? Increasing a stretches the sphere in the x direction and decreasing it squashes the sphere.

This is an *ellipsoid*, and the constants a, b, and c tell us how to graph it.

Similar considerations apply to the other surfaces.

12. Functions on Euclidean Spaces

Functions of the form $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$, are called *real valued functions in n variables.* The coordinates of points in U are the variables of which we speak. Thus if $P = (x, y, z)$, we may write $f(x, y, z)$ instead of $f(P)$. These mean the same thing.

As before, U is the domain of f, $\mathbb{R}$ is the codomain, and $f[U]$ is the range. If the function is given by a rule and the domain is not specified, the domain is assumed to be the largest set of points where the rule makes sense.

Example 11. Let $f(x, y, z) = \frac{1}{\sqrt{16 - x^2 - y^2 - z^2}}$. Find the domain of f.

Solution. This function makes sense when $\sqrt{16 - x^2 - y^2 - z^2}$ is defined and nonzero. It is defined and nonzero when $16 - x^2 - y^2 - z^2 > 0$, i.e., when $x^2 + y^2 + z^2 < 16$. This happens within an open ball of radius 4 about the origin. $\square$

Note that to graph a function $f : \mathbb{R} \to \mathbb{R}$, we use one dimension for the domain and one for the range. Thus the number of dimensions we need to draw both the domain and range, and the relationship between them, is the number of dimensions of the domain plus the number of dimensions of the range, that is, the dimension of the graph of f.

We know how to draw 3 dimensions, so for functions $f : \mathbb{R}^2 \to \mathbb{R}$, this works out. Merely let $z = f(x, y)$, that is, plot the set of points $(x, y, f(x, y))$.

If f is smooth, the set of all points $(x, y, f(x, y))$ form a surface in $\mathbb{R}^3$.

Note that the paraboloid $f(x, y) = z = x^2 + y^2$ is a function. The sphere $x^2 + y^2 + z^2 = 1$ is not a function, but its upper hemisphere $f(x, y) = z = \sqrt{1 - x^2 - y^2}$ is.

Example 12. Graph $f(x, y) = \cos(x^2 + y^2)$.

Solution. The definition makes sense for all points in $\mathbb{R}^2$, so this is the domain for f. The maximum value of f is 1 and its minimum value is -1, so the range of f is $[-1, 1]$.

Consider a circle of radius a from the origin. For (x, y) on this circle, we have $x^2 + y^2 = a^2$. Thus $f(x, y) = \cos a^2$ for all such points. The graph moves outwards in ripples, with $f(0, 0) = 1$.

The last point to consider is the distance between the cusps of the ripples. Is this distance static, increasing, or decreasing as we move away from the origin?

For $a > 1$, as a increases, a^2 increases even faster. Thus the cosine whips through all angles from 0 to $2\pi \pmod{2\pi}$ faster and faster. Therefore the distance between the cusps is decreasing, and the ripples get closer together as we move away from the origin. $\square$

Now consider $f : \mathbb{R}^3 \to \mathbb{R}$. The sum of the dimensions is four, and we have no way to depict both the domain and the range simultaneously in a graph.

Instead we do the next best thing.

Let $f : A \to B$ be a function. A *level set* of A is the preimage of a single point in B.

Specifically, if $f : \mathbb{R}^2 \to \mathbb{R}$ and $K \in \mathbb{R}$, the *level curve of f at K* is the subset of the $\mathbb{R}^2$

$$f^{-1}(K) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = K\}.$$

If we already have a graph of f, we obtain level curves by slicing the surface with a plane parallel to the xy-plane, obtaining a curve in $\mathbb{R}^3$ called a *contour curve*, and projecting the curve of intersection onto the xy-plane, which we think of as the domain, obtaining the level curve in $\mathbb{R}^2$. A number of different level curves gives us a feel for the graph.

If $f : \mathbb{R}^3 \to \mathbb{R}$ and $K \in \mathbb{R}$, the *level surface of f at K* is the subset of $\mathbb{R}^3$

$$f^{-1}(K) = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = K\}.$$

Level surfaces are even more useful than level curves, since they are virtually the only means we have of visualizing the function.

Example 13. Draw some level surfaces for the function $f(x, y, z) = x^2 + y^2 - z^2$. $\square$

There is one way to imagine the fourth dimension, and that is time. Thus let $f(x, y, z) = t$, with t being time, and imagine the level surfaces of f moving smoothly from one to the next as time passes. In this way, the "graph" of a function $f : \mathbb{R}^3 \to \mathbb{R}$ becomes a motion picture.

13. Limits and Continuity

We have already discussed limits and continuity in general. Here are a few caveats about function $f : \mathbb{R}^n \to \mathbb{R}$.

First note that analogous limit and continuity laws apply here as they did for real-valued functions of a single variable. For example, the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, et cetera. Also, the sum, product, and composition of continuous functions is continuous. Thus all polynomials in multiple variables are continuous, and rational functions are continuous in their domains.

For functions $f : \mathbb{R} \to \mathbb{R}$, if we wish to find its limit at a point x_0, we approach x_0 from the left and from the right, and see if we get the same value.

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$, and we wish to find $\lim_{(x,y)\to(x_0,y_0)}$. There are infinitely many paths to take for an approach. If all of them give the same limit, the limit of the function exists; if any two of them are different, the limit does not exist.

Example 14. Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Approach along the lines $y = 0$ and $x = 0$ and get different limits. □

Example 15. Let $f(x,y) = \frac{xy}{x^2+y^2}$. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Approach along x or y axis gives a limit of zero, but along the line $y = mx$ gives $\frac{m}{m^2+1}$. □

Example 16. Let $f(x,y) = \frac{xy^2}{x^2+y^4}$. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Approach along the lines $y = 0$, $x = 0$, $y = x$ and get the same answer. But for $x = y^2$, get a different answer. (See Stewart pg 726 for graph.) □

However, to show that a limit does exist, testing infinitely many paths is not an approach which is feasible. Instead, we use limit laws to reduce to cases of functions which we know to be continuous (or not). The limit exists at every point of continuity and is equal to the value of f.

Example 17. Let $f(x,y) = \frac{x^3-xy^2}{x^2+y^2}$. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution. Try the lines $y = 0$, $x = 0$, and $y = x$. These give the same answer. Try the curve $y = x^2$. This gives the same answer.

We begin to suspect that the limit exists. Simplify f - pull an x out of the numerator to get $f(x,y) = x$ when $(x,y) \neq (0,0)$. Thus the limit is 0. □

14. Partial Derivatives

In calculus of a single variables, the derivative of a function at a point can be interpreted as the slope of the tangent line of the curve which is the graph of the function. The slope of this line plus the point $(x, f(x))$ which is on the line allow us to compute the entire line. Since the line approximates the curve, this allows us to approximate the function near x.

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the graph is a surface, not a curve, and the natural tangential object is the tangent plane, which in turn contains many tangent lines. If we know the slope of just two of these lines, and we know value of $f(x,y)$, then we know the point $(x, y, f(x,y))$ and we will be able to determine the entire plane.

The natural lines on the tangent plane to examine are those whose projections onto the xy-plane are parallel to the axes.

Let (x_0, y_0) be a point in the domain of f. Now if we fix one of the variables, say $x = x_0$, and let y vary, we obtain a curve on the surface which is the graph of a function of a single variable. We define the *partial derivative* of $f(x,y)$ to be the derivative of this function.

Thus the partial derivatives of $f(x,y)=z$ with respect to x and y at the point (x,y) are

$$\frac{\partial z}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h};\qquad \frac{\partial z}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.$$

By letting x and y vary, the partial derivatives become functions in their own right.

We may also write $f_x(x,y)$ instead of $\frac{\partial z}{\partial x}(x,y)$ and $f_y(x,y)$ instead of $\frac{\partial z}{\partial x}(x,y)$.

We compute partial derivatives just as they are defined: treating one variable as a constant.

Example 18. Let $f(x,y)=z=x^2+2xy^2+3x^2y^3+y$. Then $\frac{\partial z}{\partial x}=2x+2y^2+6xy^3$ and $\frac{\partial z}{\partial y}=4xy+9x^2y^2+1$. $\square$

Partial derivatives for functions $f:\mathbb{R}^n\to\mathbb{R}$ are defined and computed similarly.

15. Higher Order Partial Derivatives

Note the *first order* partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are functions from $\mathbb{R}^2$ into $\mathbb{R}$. Thus they we may take their partial derivatives. We obtain four *second order* partial derivatives in $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\partial y}$, $\frac{\partial^2 z}{\partial y\partial x}$, and $\frac{\partial^2 z}{\partial y^2}$.

Proposition 4. *If* $\frac{\partial^2 z}{\partial x\partial y}$ *and* $\frac{\partial^2 z}{\partial y\partial x}$ *exist and are continuous, then*

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}.$$

We may take third order and fourth order partial derivatives, et cetera. A function $f:\mathbb{R}^n\to\mathbb{R}$ is called *smooth* if all partial derivatives of any order exist.

16. Tangent Planes

A plane is tangent to a surface in $\mathbb{R}^3$ at the point P if:

(1) they intersect at the point P;
(2) there exists a real number $\epsilon>0$ such that this is the only point of intersection within ϵ of P;
(3) no other plane satisfies both (1) and (2). Thus a tangent plane, if it exists, is unique.

A line is tangent to a surface at the point P if:

(1) they intersect at the point P;
(2) the line lies on a tangent plane.

A vector is tangent to a surface at the point P if it represents an arrow which lies on the tangent plane.

A normal vector of a tangent plane is perpendicular to all of the vectors which are tangent to the plane. Since our surface is in $\mathbb{R}^3$ (as opposed to a higher dimension), the normal vector is unique up to a scalar multiple.

The normal vector of a surface at a point is the normal vector of the tangent plane of the surface at that point. This normal vector is perpendicular to a tangent vector at the point of any curve which lies on the surface and passes through the point. Actually, any vector tangent to the surface may be expressed as the velocity vector of a path.

In fact, the tangent plane at P_0 is the union of the vectors $P_0 + \vec{v}$, where $\vec{v}$ is such a tangent vector. We have already seen that, on a sphere, the tangent vectors are perpendicular to the position vectors. Thus, on a sphere, the position vector is a normal vector.

Example 19. Find the plane which is tangent to the sphere $x^2 + y^2 + z^2 = 1$ at the point $P_0 = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.

Solution. In this case, the normal vector is the position vector, or any scalar multiple thereof. Thus let $\vec{n} = \langle 1, 1, 1\rangle$. So our plane is $(P - P_0) \cdot \vec{n} = 0$, which simplifies to

$$x + y + z = \sqrt{3}.$$

□

However, it is not true for surfaces in general that the tangent plane is perpendicular to the position vector.

Any plane which is not parallel to the z-axis may be expressed as a function of x and y in the form

$$z = ax + by + c$$

. The intersection of this plane with a plane $y = y_0$ parallel to the xz-plane is the line in that plane $z = ax + (by_0 + c)$. Thus the slope of this line is a; we call a the slope of the plane in the x direction. Similarly, b is the slope of the plane in the y direction.

Suppose the plane which is the graph of a function $g(x, y) = ax + by + c$ is tangent to a surface which is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$. Then g is the best linear approximation of f near the point (x_0, y_0) in the domain.

Example 20. Find the plane which is tangent to the curtain $z = 25 - y^2$ at the point $P_0 = (0, 4, 9)$.

Solution. Notice that the tangent plane has no slope in the x-direction. Thus we merely need to find the slope in the y-direction. This is $\frac{dz}{dy} = -2y$, which equals -8 at our point. Extrapolation yields that the z-intercept is at $z = 41$. So the plane is $z = -8y + 41$. □

We wish to find the equation for the plane which is tangent to a surface which is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point on the surface (x_0, y_0, z_0). Note that $z_0 = f(x_0, y_0)$. We can find a normal vector to the plane if we can find two vectors which lie in the plane.

Consider the curve given by intersecting the plane $y = y_0$ with the surface. This curve is then parametrized on the variable x by $\vec{a}(x) = \langle x, y_0, f(x, y_0)\rangle$. Its tangent vector is $\vec{a}'(x) = \langle 1, 0, \frac{\partial z}{\partial x}\rangle$. Similarly $\langle 0, 1, \frac{\partial z}{\partial y}\rangle$ is a tangent vector on the plane. Call these vectors $\vec{v}$ and $\vec{w}$ respectively.

Then $\vec{n} = \vec{v} \times \vec{w} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\rangle$ is a normal vector to our plane.

Then the tangent plane is

$$((x, y, z) - (x_0, y_0, z_0)) \cdot \vec{n} = 0,$$

which simplifies to

$$z = z_0 + \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0).$$

Note the similarity between this formula and the slope of the tangent line from single variable calculus

$$y = y_0 + \frac{dy}{dx}(x - x_0).$$

Example 21. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution. Find the partials: $\frac{\partial z}{\partial x} = 4x$ and $\frac{\partial z}{\partial y} = 2y$. These are functions, so we plug in our point $(1, 1)$ from the domain: $\frac{\partial z}{\partial x}(1, 1) = 4$ and $\frac{/pz}{\partial x}(1, 1) = 2$. Then the plane is $z = 3 + 4(x - 1) + 2(y - 1)$, or $4x + 2y - z = 3$. □

17. Differentials

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be injective and smooth. The graph of such a function is a set of dimension n lying inside $\mathbb{R}^{n+1}$.

Intuitively, a function $f : \mathbb{R}^n \to \mathbb{R}$ is *differentiable* when its graph may be locally approximated by a flat set of dimension n. In the case $f : \mathbb{R}^2 \to \mathbb{R}$, this flat set is a tangent plane. Let $(x_0, y_0) \in \mathbb{R}^2$ and let $z_0 = f(x_0, y_0)$. The tangent plane of f at (x_0, y_0) is also a function, say $g : \mathbb{R}^2 \to \mathbb{R}$. Then $g(x, y) = z_0$, and for a point (x, y) near (x_0, y_0), $f(x, y) \doteq g(x, y)$.

Let $\Delta x = x - x_0$, $\Delta y = y - y_0$, and $\Delta z = f(x, y) - z_0$. Then Δz is the change in height as we move along the surface from the point over (x_0, y_0) to the point over (x, y). Let $dz = g(x, y) - z_0$. Then dz is the change in height as we move along the tangent plane from the point over (x_0, y_0) to the point over (x, y).

This is true no matter what path we choose to move along the tangent plane. In particular, if first we move along a straight line in the x direction, we have a change of height equal to the slope in that direction times the distance we moved in the x direction; this is $\frac{\partial z}{\partial x}\Delta x$. From here, we move in the y direction for another change of height equal to $\frac{\partial z}{\partial y}\Delta y$. The total change in height gives us

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

We call dz the *differential* of f at (x_0, y_0). Note that dz is a function of Δx and Δy.

Example 22. Estimate $\sqrt{(2.95)^2 + (4.05)^2}$.

Solution. Let $f(x, y) = \sqrt{x^2 + y^2}$. Then

$$\frac{\partial f}{\partial x} = \frac{y}{\sqrt{x^2 + y^2}}; \qquad \frac{\partial f}{\partial y} = \frac{x}{\sqrt{x^2 + y^2}}.$$

Now let $x_0 = 3$ and $y_0 = 4$ so that $z_0 = f(3, 4) = 5$. Then

$$dz = \frac{\partial f}{\partial x}(3, 4)\Delta x + \frac{\partial f}{\partial y}(3, 4)\Delta y = \frac{3}{5}(-0.05) + \frac{4}{5}(0.05) = -\frac{3}{100} + \frac{4}{100} = \frac{1}{100}.$$

So $\sqrt{(2.95) + (4.05)} = z_0 + \Delta z \doteq z_0 + dz = 5.01$. My calculator gives 5.0105. □

18. Parametrized Surfaces

The image of a function $\vec{s} : U \to \mathbb{R}^3$, where $U \subset \mathbb{R}^2$, is called a *parametrized surface*. We think of such a function as taking a piece of the plane and placing it somewhere in space. We will not study these here, but merely mention a couple of examples.

Example 23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ so that the graph of f is a surface in $\mathbb{R}^3$. Now let $\vec{s} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\vec{s}(x, y) = \langle x, y, f(x, y)\rangle$. The image of $\vec{s}$ is the graph of f.

Example 24. Note the the unit sphere, which is the locus of the equation $x^2+y^2+z^2 = 1$, and therefore is the preimage of 1 under the function $f(x, y, z) = x^2+y^2+z^2$, cannot be expressed as the graph of a function $g : D \subset \mathbb{R}^2 \to \mathbb{R}$; this is because it is not "well-defined" for any point in the unit disk, as we have two choices for where to send the point (one above it and one below it).

However, the unit sphere may be expressed as the image of a function. As we will see when we look at spherical coordinates, the unit sphere is the image of the function $\vec{s} : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ given by

$$\vec{s}(\theta, \phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle.$$

19. Summary

Graph Curve

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then the graph of f is a curve in $\mathbb{R}^2$ which is a parabola. The picture of such a curve clearly displays both the domain and range and the relationship between them. However, not every curve in $\mathbb{R}^2$ may be expressed as the graph of a function. A tangent vector of such a curve at the point $(x, f(x))$ may be given by $\langle 1, f'(x)\rangle$.

Image Curve

Let $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ be given by $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. Then the image of $\vec{r}$ is a curve in $\mathbb{R}^3$ which is a helix. The picture of such a curve displays only the range and not the domain. Thus the picture of the curve does not give us any information about the manner in which the curve is parametrized. A tangent vector of such a curve at the point $\vec{r}(t)$ is $\vec{r}\,'(t)$.

Preimage Curve

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = y^2 - x^2$. Then the preimages of nonzero points in $\mathbb{R}$ under f are level curves of f in $\mathbb{R}^2$ which are hyperbolas. The picture of such a curve displays only the domain of the function. We must draw many level curves in the domain to get an idea of the function itself. To find the tangent vector, we may restate the curve as an image (parametrize the curve) or as a graph (solve for y), or manipulate the gradient vector.

Graph Surface

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = \sqrt{9 - x^2 - y^2}$. Then the graph of f is a surface in $\mathbb{R}^3$ which is a hemisphere. The picture of such a surface displays both the domain and range and the relationship between them. The normal vector of such a surface at the point $(x, y, f(x, y))$ is $\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\rangle$.

Image Surface

Let $\vec{s} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\vec{s}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$. Then the image of $\vec{s}$ is a surface in $\mathbb{R}^3$ which is a cone. The picture of such a surface displays only the range. The picture alone does not indicate how the surface is parametrized. The normal vector for such a surface may be computed without too much difficulty, but we will not derive it in this course.

Preimage Surface

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 - z^2$. Then the preimage of a point K in $\mathbb{R}$ under f is a level surface of f in $\mathbb{R}^3$ which is a one sheeted hyperboloid (if $K > 0$), a cone (if $K = 0$), or a two sheeted hyperboloid (if $K < 0$). The picture of such a surface displays only the domain, and to obtain information about the function, we may draw many level surfaces in the same picture. The normal vector for such a surface is given by the gradient.

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