VECTOR CALCULUS
INTEGRATION OF SCALAR FIELDS

PAUL L. BAILEY

ABSTRACT. These notes are copied from lecture notes given at University of California, Irvine. While we have covered much of this, we now have the idea of the Jacobian to aid in our understanding of change of coordinates, the higher dimensional analog of substitution.

1. Review of Riemann Integration

Let $f : [a, b] \to \mathbb{R}$, where $[a, b]$ is a closed interval of real numbers. A *partition* of $[a, b]$ given by numbers $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is the set of intervals $I_i = [x_{i-1}, x_i]$;

$$\mathcal{P} = \{I_i \mid i = 1, \dots, n\}$$

.

The *mesh* of the partition is length of the largest interval in it and is denoted $|\mathcal{P}|$.

Let $\Delta x_i = x_i - x_{i-1}$ be the length of the interval I_i. Let $x_i^* \in I_i$ be any point in the interval. The *Riemann sum* of f over $\mathcal{P}$ is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

The *Riemann integral* of f on $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{|P|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

provided that this limit exists and is independent of the partitions and x_i^* chosen. If f is continuous, this provision is always satisfied.

Date: April 19, 2018.

2. Double Integration over Rectangular Regions

Suppose we have a real valued function f defined on a rectangular region in $\mathbb{R}^2$. We wish to calculate the signed volume of the region in $\mathbb{R}^3$ which lies between the graph of f and the xy-plane.

Here we use the standard calculus trick. Break up the domain into smaller rectangles. Pick some point in the rectangle. Evaluate the function at that point and multiply this value by the area of the small rectangle. This gives the volume of a small column. The sum of these volumes approximates the volume under the surface, and this approximation becomes exact as the rectangle become infinitesimally small.

Let $D \subset \mathbb{R}^2$ be a rectangular domain $D = [a, b] \times [c, d]$. A *partition* $\mathcal{P}$ of D into rectangular subdomains is given by partitions of $[a, b]$ and $[c, d]$ into intervals:

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b;$$

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d;$$

so that the $(i, j)^{\text{th}}$ rectangle is

$$R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

and the partition is the set of all such rectangles

$$\mathcal{P} = \{R_{i,j} \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}.$$

The *mesh* of $\mathcal{P}$ is

$$|\mathcal{P}| = \max\{|(x_i, y_j) - (x_{i-1}, y_{j-1})| : i = 1, \ldots, m \text{ and} j = 1, \ldots, n\},$$

the maximum diagonal of any rectangle in the partition.

Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$ be the lengths of the sides of the rectangle $R_{i,j}$ so that its area is $\Delta A_{i,j} = \Delta x_i \Delta y_j$.

Let (x_i^*, y_j^*) be an arbitrary point in $R_{i,j}$. Define the *Riemann sum* of f over $\mathcal{P}$ to be

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{i,j}.$$

Define the *Riemann integral* of f over D to be

$$\iint_D f(x, y) dA = \lim_{|P| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{i,j},$$

provided that this limit exists and is independent of the partitions and the points (x_i^*, y_j^*) chosen. If f is continuous, this provision is always satisfied.

Proposition 1. Fubini's Theorem
Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$. If f is continuous, then its Riemann integral exists and

$$\iint_D f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy.$$

Proof. The basic idea behind this is as follows. The region under the graph of the function is a solid in $\mathbb{R}^3$.

For each $y^* \in [a, b]$, the value

$$A(y^*) = \int_c^d f(x, y^*)dx$$

is the area of the slice of the solid obtained by intersecting the plane $y = y^*$ with the solid.

Partition the interval $[a, b]$ into m subintervals of equal length Δy. In each subinterval, select a representative y_i^*. The volume of the solid is approximated by the sum of the volumes of the slabs

$$V \doteq \sum_{i=1}^{m} A(y_i^*)\Delta y.$$

This approximation becomes exact as $\Delta y \to 0$. □

Integrals of the form $\int_a^b \int_c^d f(x, y)dydx$ are called *iterated integrals*, since the integration is an iterated process; the first iteration is with respect to y, and the second is with respect to x.

Proposition 2. Properties of Double Integration
Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous. Let $k \in \mathbb{R}$. Then

(1) $\iint_D (f(x, y) + g(x, y))dA = \iint_D f(x, y)dA + \iint_D g(x, y)dA$;
(2) $\iint_D kf(x, y)dA = k \iint_D f(x, y)dA$;
(3) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in D \Rightarrow \iint_D f(x, y)dA \leq \iint_D f(x, y)dA$.

Proposition 3. *Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous. Suppose that f is a function only of x and g is a function only of y. Then*

$$\iint_D f(x)g(y)dA = \int_a^b f(x)dx \int_c^d g(y)dy.$$

Proposition 4. *Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be continuous. Suppose $f(x, y) \geq 0$ for every $(x, y) \in D$. Then*

$$\iint_D f(x, y)dA = 0 \Rightarrow f(x, y) = 0 \text{ for all } (x, y) \in D.$$

3. Double Integration on Bounded Regions

Let $D \subset \mathbb{R}^2$. We say that D is *bounded* if the is a rectangle R such that $D \subset R$.

Let $f : D \to \mathbb{R}$, where D is bounded by the rectangle R. Define a function $f_R : R \to \mathbb{R}$ by

$$f_R(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \ ; \\ 0 & \text{otherwise} \ . \end{cases}$$

Then we define

$$\iint_D f(x,y)dA = \iint_R f_R(x,y)dA.$$

Only the portion of f_R which lies over the region D contributes to the integral.

If we break up D into smaller pieces such that the pieces overlap only on their boundaries, these pieces are said to be *internally disjoint*.

Proposition 5. *Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$. Suppose D is the union of internally disjoint pieces $D_1, \dots, D_n$. Then*

$$\iint_D f(x,y)dA = \sum_{i=1}^{n} \iint_{D_i} f(x,y)dA.$$

4. Review of Substitution

In calculus of a single variable, some functions are more easily integrated if the coordinate system of their domain is modified. This is accomplished by "pulling back" the domain into another domain; that is, by making the domain be the range of a transformation.

For example, let $f(x) = \cos(x^2)$. Since the cosine of x^2 is more complex than the cosine of u, we wish to view this function instead as $f(x(u)) = \cos u$; thus we are now considering a composite function, the first transforms u into $x = \sqrt{u}$, and the second takes the cosine of $x^2 = \sqrt{u}^2 = u$.

We partition the interval where u resides into subintervals and select a point u_i^* in each. Now $x(u_i^*)$ is in the domain of f. A Riemann sum for f is

$$\sum f(x(u_i^*))\Delta x.$$

What is Δx in terms of u? It is approximately $\frac{dx}{du}\Delta u$, and this approximation improves as $\Delta u \to 0$.

Thus our integral becomes

$$\begin{aligned}\int f(x)dx &= \lim_{\Delta x \to 0} \sum f(x_i^*)\Delta x \\ &= \lim_{\Delta u \to 0} \sum f(x(u_i^*))\frac{dx}{du}\Delta u \\ &= \int f(x(u))\frac{dx}{du}du.\end{aligned}$$

The factor $\frac{dx}{du}$ is the *Jacobian*, or *distortion factor* caused by the change of coordinates; it is the amount that the transformation $u \mapsto x(u)$ warps lengths.

In the example above, $x = \sqrt{u}$ so $\frac{dx}{du} = \frac{1}{2\sqrt{u}}$. Thus $\int 2x\cos(x^2)dx = \int 2\sqrt{u}\cos(u)\frac{1}{2\sqrt{u}}du = \int \cos(u) = \sin(u) = \sin(x^2)$.

As another example, let $f(x) = \sqrt{1 + x^2}$. The graph of this function is the part of the hyperbola $x^2 - y^2 = 1$ which lies above the line $y = 0$. We wish to find $\int_1^2 f(x)dx$. We use the transformation $x(u) = \sec u$. Think of this as a change of coordinate systems on a line.

In terms of u instead of x, the function f is more easily understood:

$$f(x(u)) = \sqrt{1 + \sec^2 u} = \sqrt{\tan^2 u} = \tan u.$$

We also pull back the interval $[1, 2]$. Note that $\sec u = 1$ when $u = 0$ and $\sec u = 2$ when $u = \pi/3$.

We look at the area under the curve $\tan u$ between $u = 0$ and $u = \pi/3$. We partition this up into subintervals and select a point from each. We then multiply the height of the curve over that point by the *distorted* length of the subinterval; that is the approximate length of the subinterval in x coordinates (in the original domain of f) which is the image under the coordinate map of our subinterval in u coordinates. This length is approximated by $\frac{dx}{du} = \tan u \sec u \Delta u$. Note that the distortion factor $\tan u \sec u$ depends on u; the coordinate transformation distorts more in some places than in others.

5. Planar Transformation Area Distortion

Let $E \subset \mathbb{R}^2$. Let $\vec{s} : E \to \mathbb{R}^2$ be a transformation of the plane. Describe the coordinates of the domain with the variables u and v and the coordinates of the range with the variables x and y. Thus $\vec{s}(u, v) = \langle x(u, v), y(u, v)\rangle$.

We say that $\vec{s}$ is *internally injective* if it is injective everywhere on D except possibly on the boundary of D. We say that $\vec{s}$ is a *coordinate transformation* if it is surjective and internally injective.

Example 1. Let $\vec{s}(u, v) = \langle 2u, 2v\rangle$. That is, the coordinate transformation of the uv-plane into the xy-plane is given by $x = 2u$ and $y = 2v$. How does this transformation distort area?

Solution. We may think of $\vec{s}$ simply as multiplying the coordinates of each point by 2. Thus $\vec{s}$ expands the entire plane by a factor of 2 in every direction. A rectangle of length l and width w is sent to a rectangle of length $2l$ and width $2w$. The original area is lw, and the resultant area is $4lw$. It is clear that $\vec{s}$ expands area by a factor of 4 throughout the plane. □

Example 2. Let $\vec{s}(u, v) = \langle au, bv\rangle$. How does $\vec{s}$ distort area?

Solution. Here $x = au$ and $v = by$, so that $u = \frac{x}{a}$ and $v = \frac{y}{b}$. This transformation distorts area by a factors of ab everywhere on the plane.

Note that this transformation sends a circle of radius r centered at origin, given by $u^2 + v^2 = r^2$ to an ellipse $\frac{x^2}{a^2r^2} + \frac{y^2}{b^2r^2} = 1$. Thus the area of the ellipse is the area of the circle times ab; that is, the area of the ellipse is $ab\pi r^2$. □

Example 3. How do polar coordinates distort area?

Discussion. The coordinate transformation for polar coordinates is $\vec{s}(r, \theta) = \langle r\cos\theta, r\sin\theta\rangle$. The domain of $\vec{s}$ is $[0, \infty) \times [0, 2\pi)$. A narrow horizontal strip in the domain is compressed to a point for $r = 0$ and is expanded like a fan pivoting on the point elsewhere. It is clear that the area is distorting to a greater extent the farther away from the origin one looks; thus the distortion factor depends on r. Exactly how it depends on r we will now discover. □

We wish to see how, in general, a coordinate transformation distorts area near a point (u_0, v_0) in the domain. Let Δu be a small change in u and let Δv be a small change in v. Consider the rectangle $[u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$, which resides in the uv-plane. The transformation $\vec{s}$ maps this rectangle onto some region in the xy-plane. We wish to estimate its area.

Create two paths which are lines in the uv-plane and whose velocity vectors at (u_0, v_0) are the sides of the rectangle. Then see where in the xy-plane these vectors get mapped by $\vec{s}$.

Let $\vec{r}_1(t) = \langle u_0 + t\Delta u, v_0\rangle$. This is a horizontal line in the uv-plane that passes through (u_0, v_0) at $t = 0$. Let $\vec{r}_2(t) = \langle u_0, v_0 + t\Delta v\rangle$. The velocity vectors of these curves at (u_0, v_0) are $\vec{r}_1'(0) = \langle \Delta u, 0\rangle$ and $\vec{r}_2'(0) = \langle 0, \Delta v\rangle$. Thus the area of the rectangle is the length of their cross product, which is $\Delta u\Delta v$.

These vectors outline the rectangle in the uv-plane. Their images under $\vec{s}$ outline the image of the rectangle in the xy-plane.

To find these image vectors, take the velocity vectors of the compositions. Let $\vec{c_1} = \vec{s} \circ \vec{r}_1$ and let $\vec{c_2} = \vec{s} \circ \vec{r}_2$. The image of $\vec{c_i}$ in the xy-plane is where $\vec{s}$ maps the line $\vec{r}_i$. We may express these as $\vec{c_1} = \langle x(u_0 + t\Delta u, v_0), y(u_0 + t\Delta u, v_0)\rangle$ and $\vec{c_2} = \langle x(u_0, v_0 + t\Delta v), y(u_0, v_0 + t\Delta v)\rangle$. Now use the chain rule to compute the velocity vectors of the transformed curves:

$$\vec{c}_1'(0) = \langle \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}, \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\rangle = \langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u\rangle;$$

$$\vec{c}_2'(0) = \langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v\rangle.$$

The area of the transformed rectangle is approximated by the area of the parallelogram determined by these velocity vectors:

$$|\vec{c}_1'(0) \times \vec{c}_2'(0)| = (\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u})\Delta u \Delta v.$$

The above expression is evaluated at the point (u_0, v_0) to obtain the distorted area. The quantity

$$J_{\vec{s}} = (\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u})$$

is called the *Jacobian* of the transformation, and represents the distortion factor of the transformation.

As Δu and Δv become smaller, our approximation improves. We call

$$dA = dxdy = J_{\vec{s}}dudv$$

the *area elements* of the coordinate systems. That is, $dxdy$ is the area element of rectangular coordinates and $Jdudv$ is the area element of the alternate uv coordinate system.

The Jacobian may be remembered as the determinant of a matrix: if $\vec{s}(u, v) = \langle x(u, v), y(u, v)\rangle$, then

$$J_{\vec{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example 4. Compute the area element of polar coordinates.

Solution. The polar coordinate transformation is $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. The partials of this transformation are $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, and $\frac{\partial y}{\partial \theta} = r\cos\theta$.

The Jacobian of the polar transformation is

$$\begin{aligned} J_{\texttt{polar}}(r, \theta) &= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \\ &= (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \\ &= r\cos^2\theta + r\sin^2\theta = r. \end{aligned}$$

Thus $dA = rdrd\theta$. The polar coordinate transformation distorts area by an amount equal to the distance in the $r\theta$-plane from the origin. □

Another interesting transformation is given by $x(u,v) = u^2 - v^2$ and $y(u,v) = 2uv$. This transformation arises naturally in the theory of functions of complex variables, and is in fact the transformation of the complex plain given by $a + ib \mapsto (a + ib)^2$. It takes the plane and wraps it around itself one time, pivoting at the origin. It has the following properties:

- the preimages of lines in the xy-plane are hyperbolas in the uv-plane;
- the images of lines not through the origin in the uv-plane are parabolas in the xy-plane;
- rays from the origin are sent to rays from the origin;
- circles centered at the origin are sent to circles centered at the origin;
- the unit circle is sent to the unit circle;
- a point with polar coordinates $(r\cos\theta, r\sin\theta)$ in the uv-plane is sent to a point with polar coordinates $(r^2\cos 2\theta, r^2\sin 2\theta)$ in the xy-plane;
- the transformation is two-to-one at every point in the xy-plane accept the origin.

Let us call this *spin coordinates.*

Example 5. The Jacobian of the spin transformation is

$$\begin{aligned} J_{\texttt{spin}}(u,v) &= \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} \\ &= 4u^2 + 4v^2. \end{aligned}$$

□

6. Double Integration via Coordinate Transformation

Let $\vec{s}(u,v) = \langle x(u,v), y(u,v)\rangle$ be a coordinate transformation. Suppose we wish to compute $\iint_D f(x,y)dA$ under this alternate coordinate system. First we pull D back from the xy-plane into the uv-plane. Let E be the preimage of D under $\vec{s}$.

Now break up E into little rectangles. There is a corresponding breakup of D into blocks of area which are the images of these rectangles.

In each rectangle, select a point (u_i^*, v_j^*). The value of the function we wish to integrate over the corresponding point in the xy-plane is $f(x(u_i^*, v_j^*))$. We then multiply this value of the area of the distorted rectangle, which approximately $\Delta A_{i,j} \doteq J\Delta u\Delta v$.

Let $x_{i,j}^* = x(u_i^*, v_j^*)$ and $y_{i,j}^* = y(u_i^*, v_j^*)$. Thus $(x_{i,j}^*, y_{i,j}^*)$ is an arbitrary element of the $(i,j)^{\text{th}}$ distorted rectangle in the xy-plane.

Thus

$$\begin{aligned}
\iint_D f(x,y)dA &\doteq \sum\sum f(x_{i,j}^*, y_{i,j}^*)\Delta A_{i,j} \\
&\doteq \sum\sum f(x(u_i^*, v_j^*, y(u_i^*, v_j^*)J_{\vec{s}}(u_i^*, v_j^*)\Delta u\Delta v \\
&\doteq \iint_E f(x(u,v), y(u,v))J_{\vec{s}}(u,v)dudv.
\end{aligned}$$

These approximations become exact as the rectangles in the uv-plane become infinitesimally small.

Example 6. Use polar coordinates to find the volume under the paraboloid $z = h^2 - x^2 - y^2$ and above the xy-plane, where h is a constant.

Solution. The domain here is a circle of radius h. In polar form, this is $r \in [0,h]$ and $\theta \in [0, 2\pi]$, and $dA = rdrd\theta$.

$$\begin{aligned}
V &= \iint_D h^2 - x^2 - y^2 dA \\
&= \int_0^{2\pi}\int_0^h (h^2 - r^2)rdrd\theta \\
&= \int_0^{2\pi} h^2\frac{r^2}{2} - \frac{r^4}{4}\Big|_0^h \\
&= \int_0^{2\pi} \frac{h^4}{4}d\theta \\
&= \frac{\pi h^4}{2}.
\end{aligned}$$

□

Example 7. Use elliptical coordinates to find the volume under the paraboloid $z = a^2b^2 - b^2x^2 - a^2y^2$ and above the xy-plane, where a and b are constants.

Solution. The the boundary of the domain is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is parametrized by $\langle a\cos\theta, b\sin\theta\rangle$. Thus let $x = au\cos\theta$ and $y = bu\sin\theta$. In this coordinate system, our domain is $u \in [0,1]$ and $\theta \in [0, 2\pi]$.

The Jacobian of the a, b-elliptical transformation is

$$\begin{aligned} J_{\texttt{ellip}}(u,\theta) &= \begin{vmatrix} a\cos\theta & b\sin\theta \\ -au\sin\theta & bu\cos\theta \end{vmatrix} \\ &= abu\cos^2\theta + abu\sin^2\theta \\ &= abu. \end{aligned}$$

Let D be the interior of the ellipse and let $E = [0,1] \times [0, 2\pi]$ in the $u\theta$-plane. Thus the volume is

$$\begin{aligned} V &= \iint_D a^2b^2 - b^2x^2 - a^2y^2 dA \\ &= \iint_E (a^2b^2 - a^2b^2u^2)(abu) du d\theta \\ &= a^3b^3 \int_0^{2\pi} \int_0^1 (u - u^3) du d\theta \\ &= a^3b^3 \frac{\pi}{2}. \end{aligned}$$

□

Example 8. Use spin coordinates to evaluate the integral $\iint_D y dA$, where D is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

Solution. Let E be the preimage of D under the spin transformation. You may verify that $E = [0,1] \times [0,1]$ in the uv-plane. The Jacobian of this transformation is $4u^2 + 4v^2$. Thus

$$\begin{aligned} \iint_D y dA &= \iint_E (2uv)(4u^2 + 4v^2) du dv \\ &= 8 \int_0^1 \int_0^1 u^3v + uv^3 du dv \\ &= 8 \int_0^1 \frac{1}{4}u^4v + \frac{1}{2}u^2v^3\Big|_0^1 dv \\ &= 8 \int_0^1 \frac{1}{4}v + \frac{1}{2}v^3 dv \\ &= 8(\frac{1}{8}v^2 + \frac{1}{8}v^4)\Big|_0^1 \\ &= 2. \end{aligned}$$

□

7. Triple Integration

Let $D = [a, b] \times [c, d] \times [p, q]$ be a rectangular box in $\mathbb{R}^3$. Let $f : D \to \mathbb{R}$. We define the triple integral of f over D as follows.

Partition each of the intervals $[a, b]$, $[c, d]$, and $[p, q]$ into smaller subintervals:

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b;$$

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d;$$

$$p = z_0 < z_1 < \cdots < z_{o-1} < z_o = q.$$

Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$.

Let $R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ be the $(ijk)^{\text{th}}$ rectangular box. The volume of R_{ijk} is $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$. The partition of D is

$$\mathcal{P} = \{R_{ijk}\}.$$

The mesh of the partition is the length of the maximum diagonal

$$|\mathcal{P}| = \max\{\sqrt{(\Delta x_i)^2 + (\Delta y_j)^2 + (\Delta z_k)^2}\}.$$

In each box, select a point (x_j^*, y_j^*, z_k^*). The Riemann sum over this partition is

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{o} f(x_i^*, y_j^*, z_o^*) \Delta V.$$

The triple integral is defined as

$$\iiint_D f(x, y, z) dV = \lim_{|P| \to 0} \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{o} f(x_i^*, y_j^*, z_o^*) \Delta V.$$

If D is not a rectangular box but is contained in a rectangular box R, we extend f to f_R by defining f_R to be zero in $R \setminus D$. We define the integral over D of f to be the integral over R of f_R.

We have Fubini's Theorem for triple integrals. That is, if f is continuous, triple integrals over rectangular regions may be evaluated as iterated integrals. There are six combinations of orders of integration:

$$dV = dxdydz = dxdzdy = dydxdz = dydzdx = dzdxdy = dzdydx.$$

Choose the order which makes the integration as easy as possible.

Example 9. Let D be the unit cube in $\mathbb{R}^3$, $D = [0,1] \times [0,1] \times [0,1]$. Suppose that the density of the cube at a given point in the cube is proportional to the height of the point. Find the mass of the cube.

Solution. Let $\mu(x, y, z) = kz$, where k is a constant. Then μ is a function which represents the density of the cube at the point (x, y, z). Thus the mass of a little subcube is approximately the density in that subcude times the volume of the subcube. To attain the total mass, we integrate the density.

Thus

$$\begin{aligned} \text{mass} &= \iiint_D \mu(x, y, z) dV \\ &= \int_0^1 \int_0^1 \int_0^1 kz dz dy dx \\ &= \int_0^1 dx \int_0^1 dy \int_0^1 kz dz \\ &= k\frac{z^2}{2}\Big|_0^1 \\ &= \frac{k}{2}. \end{aligned}$$

□

8. Spatial Transformation Volume Distortion

Let $E \subset \mathbb{R}^3$. Let $\vec{s} : E \to \mathbb{R}^3$ be a transformation of the space. Describe the coordinates of the domain with the variables u, v, and w and the the coordinates of the range with the variables x, y, and z. Thus

$$\vec{s}(u, v, w) = \langle x(u, v, w), y(u, v, w), z(u, v, w) \rangle.$$

We wish to see how such a transformation distorts volume near a point (u_0, v_0, w_0) in the domain. Let Δu be a small change in u, Δv a small change in v, and Δw a small change in w. Consider the rectangular box $[u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v] \times [w_0, w_0 + \Delta w_0]$ which resides in uvw-space. The transformation $\vec{s}$ maps this rectangular box onto some region in xyz-space. Think of this region as a distortion of the original box. We wish to estimate the volume of the distorted box.

As we did before, parametrize lines through the point (u_0, v_0, w_0) which are parallel to the coordinate axes and which have velocity vectors $\langle \Delta u, 0, 0 \rangle$, $\langle 0, \Delta v, 0 \rangle$, and $\langle 0, 0, \Delta w \rangle$. The volume of the little box in uvw-space is the triple scalar product of these vectors, which is $\Delta u \Delta v \Delta w$.

The volume of the distorted box which is the image of our little rectangular box under the change of coordinate map $\vec{s}$ is approximated by the triple scalar product of the images of these vectors. These image vectors, by the chain rule, are $\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, \frac{\partial z}{\partial u}\Delta u \rangle$, $\langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, \frac{\partial z}{\partial v}\Delta v \rangle$, and $\langle \frac{\partial x}{\partial w}\Delta w, \frac{\partial y}{\partial w}\Delta w, \frac{\partial z}{\partial w}\Delta w \rangle$.

Define the Jacobian

$$J_{\vec{s}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The volume spanned by the vectors above is then

$$\Delta V = J_{\vec{s}}(u_0, v_0, w_0)\Delta u \Delta v \Delta w.$$

Again, as in the planar case, this matrix of partials called the Jacobian matrix represents the linearization of the transformation near the point (u_0, v_0, w_0) and its determinant, called the Jacobian, represents the distortion factor of the transformation.

Example 10. Find the Jacobian of the cylindrical coordinate transformation.

Solution. The cylindrical transformation is given by

$$(r, \theta, z) \mapsto \langle r\cos\theta, r\sin\theta, z\rangle.$$

Since $\frac{\partial x}{\partial z} = \frac{\partial y}{\partial z} = 0$, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial \theta} = 0$, and $\frac{\partial z}{\partial z} = 1$, this determinant is actually just the determinant of the polar coordinate transformation. Thus the Jacobian is

$$J_{\texttt{cylin}} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & 0 \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = r.$$

□

Example 11. Find the Jacobian of the spherical coordinate transformation.

Solution. The spherical transformation is given by

$$(\rho, \theta, \phi) \mapsto \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle.$$

Thus the Jacobian is

$$J_{\texttt{sphere}} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \end{vmatrix} = \rho^2\sin\phi.$$

□

9. Triple Integration via Coordinate Transformation

Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$. Label the coordinates in D as x, y and z.

Let $\vec{s} : E \subset \mathbb{R}^3 \to D$ be a coordinate transformation. Label the coordinates in E as u, v and w. Note that E is the preimage of D under $\vec{s}$.

For the same reasons as in the planar case, the integral transforms as

$$\iiint_D f(x,y,z)dV = \iiint_E f(x(u,v,w), y(u,v,w), z(u,v,w))J_{\vec{s}}(u,v,w)dudvdw.$$

Example 12. The density of a ball of radius R centered at the origin is given by the function $\mu(x,y,z) = \frac{1}{x^2+y^2+z^2+1}$. Find the mass of the ball.

Solution. We integrate using spherical coordinates. Let D be the ball. The ball is the image of $E = [0,R] \times [0,2\pi] \times [0,\pi]$ in $\rho\theta\phi$-space. The Jacobian for the spherical transformation is $\rho^2 \sin\phi$. Thus

$$\begin{aligned}
M &= \iiint_D \mu(x,y,z)dV \\
&= \iiint_E \mu(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))Jd\rho d\theta d\phi \\
&= \int_0^\pi \int_0^{2\pi} \int_0^R \frac{1}{1+\rho^2}\rho^2 \sin\phi d\rho d\theta d\phi \\
&= \int_0^\pi \sin\phi d\phi \int_0^{2\pi} d\theta \int_0^R \frac{\rho^2}{1+\rho^2} d\rho \\
&= -\cos\phi\Big|_0^\pi \theta\Big|_0^{2\pi} \int_0^R \frac{1+\rho^2}{1+\rho^2} - \frac{1}{1+\rho^2} d\rho \\
&= 4\pi(\rho - \arctan\rho)\Big|_0^R \\
&= 4\pi(R - \arctan R).
\end{aligned}$$

□

BASIS Scottsdale
E-mail address: paul.bailey@basised.com