CATEGORY THEORY
SEMIDIRECT PRODUCT

PAUL L. BAILEY

1. Normalizers

Definition 1. Let G be a group, $X \subset G$, and $g \in G$. Set
$$gXg^{-1} = \{y \in G \mid y = gxg^{-1} \text{ for some } x \in X\}.$$
We say that g *normalizes* X if $gXg^{-1} = X$ for all $x \in X$.

Let $A \subset G$. We say that A normalizes X if a normalizes X for every $a \in A$.

Let $H \leq G$. The *normalizer* of X in H is
$$N_H(X) = \{h \in H \mid hXh^{-1} = X\}.$$

Proposition 1. *Let G be a group, $H \leq G$, and $X \subset G$. Then $N_H(X) \leq H$.*

Proof. Since $1 \in H$, and $1x1^{-1} = x$ for all $x \in X$, we know $1 \in N_H(X)$.

Let $h_1, h_2 \in N_H(X)$. Then
$$h_1h_2X(h_1h_2)^{-1} = h_1(h_2Xh_2^{-1})h_1^{-1} = h_1Xh_1^{-1} = X,$$
so $h_1h_2 \in N_H(X)$.

Let $h \in H$, so that $X = hXh^{-1}$. Multiply both sides by h^{-1} on the left and by h on the right to get
$$h^{-1}Xh = h^{-1}hXh^{-1}h = X;$$
thus $h^{-1} \in N_H(X)$. □

Observation 1. If we define the *subnormalizer* of X in H to be
$$\widehat{N}_H(X) = \{h \in H \mid hXh^{-1} \subset X\} = \{h \in H \mid hxh^{-1} \in X \text{ for all } x \in X\},$$
we wonder if these definitions are equivalent. Our proof of closure under inverses relied on the first definition. Clearly $\widehat{N}_H(X) \subset N_H(X)$, but are these sets equal?

We found a counterexample (StackExchange Mathematics Question # 2413065):

Take $G = \mathrm{Sym}(\mathbb{Z})$, the group of permutations of the integers, and let X be the subgroup of permutations that leave the natural numbers fixed. Let g be the permutation $m \mapsto m+1$. Then $g \in \widehat{N}_G(X)$: indeed, if $\sigma \in X$, then for all $n \geq 0$ we have $(g^{-1}\sigma g)(n) = \sigma(n+1) - 1 = n$ because $n+1 \geq 0$. But $g^{-1} \notin \widehat{N}_G(X)$, because if we take $\tau = (-1\ \ {-2}) \in X$, we get $(g\tau g^{-1})(0) = \tau(0-1)+1 = -2+1 = -1$, so $g\tau g^{-1} \notin X$.

Date: Monday, November 11, 2019.

Here is another example: `https://math.stackexchange.com/posts/107866/edit`

Consider the group of matrices

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x \in \mathbb{Q}^\times, y \in \mathbb{Q} \right\} = \mathrm{AGL}(1, \mathbb{Q})$$

and its subgroup

$$H = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} : y \in \mathbb{Z} \right\} \cong \mathbb{Z}$$

and of course the single element

$$a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

A direct calculation gives

$$aHa^{-1} = \left\{ \begin{bmatrix} 1 & 2y \\ 0 & 1 \end{bmatrix} : y \in \mathbb{Z} \right\} < H$$

is a proper subgroup of *H*.

2. Products

Definition 2. Let G be a group and let $X, Y \subset G$. Set

$$XY = \{xy \in G \mid x \in X \text{ and } y \in Y\} \quad \text{and} \quad X^{-1} = \{x^{-1} \in G \mid x \in X\}.$$

Proposition 2. *Let G be a group and let $H, K \leq G$. Then $HK \leq G$ if and only if $HK = KH$.*

Proof. If $M \leq G$, then $M^{-1} = M$. Thus if $HK \leq G$, then $HK = (HK)^{-1} = K^{-1}H^{-1} = KH$.

Suppose $HK = KH$. Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$ so that h_1k_1 and h_2k_2 are arbitrary members of HK. Since $HK = KH$, there exists $k_3 \in K$ such that $k_1h_2 = h_2k_3$. Then $h_1k_1h_2k_2 = h_1h_2k_3k_2 \in HK$.

Let $h \in H$ and $k \in K$ so that hk is an arbitrary member of HK. Then $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$. Thus $HK \leq G$. $\square$

3. Internal Direct Product

Proposition 3. *Let G be a group and let $H, K \leq G$ with $K \lhd G$. Then $KH \leq G$.*

Proof. Let k_1h_1, k_2h_2 be arbitrary elements of KH, where $k_1, k_2 \in= K$ and $h_1, h_2 \in K$. Then

$$k_1h_1k_2h_2 = k_1h_1k_2h_1^{-1}h_1h_2 = k_1k_3h_1h_2,$$

where $k_3 = h_1k_2h_1^{-1} \in K$ since K is normal in G. Since $k_1k_3 \in K$ and $h_1h_2 \in H$, we have $k_1h_1k_2h_2 \in KH$.

Let $kh \in KH$ where $k \in K$ and $h \in H$. Then $(kh)^{-1} = h^{-1}k^{-1} = h^{-1}k^{-1}hh^{-1} = k_0h^{-1}$, where $k_0 = h^{-1}k^{-1}h \in K$ since $K \lhd G$. Thus $(kh)^{-1} \in KH$.

Therefore $KH \leq G$. □

Proposition 4. *Let G be a group and let $H, K \lhd G$. Then*

(a) *$H \cap K \lhd G$;*
(b) *$HK \lhd G$.*

Proof. Recall that to show $N \lhd G$, it suffices to show that $gng^{-1} \in N$ for all $n \in N$ and all $g \in G$.

(a) Let $h \in H \cap K$ and $g \in G$. Then $g^{-1}hg \in H$ since $H \lhd G$ and $h \in H$. Also $g^{-1}hg \in K$ since $K \lhd G$ and $h \in K$. Thus $g^{-1}hg \in H \cap K$, so $H \cap K \lhd G$.

(b) Let $hk \in HK$, where $h \in H$ and $k \in K$, and let $g \in G$. Then $g^{-1}hkg = g^{-1}hgg^{-1}kg = h_0k_0$, where $h0 = g^{-1}hg \in H$ and $k_0 = g^{-1}kg \in K$. Thus $g^{-1}hkg \in HK$, so $HK \lhd G$. □

Proposition 5. *Let G be a group and let H and K be normal subgroups of G which intersect trivially. Then the elements of H commute with the elements of K.*

Proof. Let $h \in H$ and $k \in K$. Since H is normal in G, $k^{-1}hk \in H$, so $h^{-1}k^{-1}hk \in H$. Since K is normal in G, $h^{-1}k^{-1}h \in K$, so $h^{-1}k^{-1}hk \in K$. Thus $h^{-1}k^{-1}hk \in H \cap K = \{1\}$, so $h^{-1}k^{-1}hk = 1$. Thus $hk = kh$. □

Proposition 6. *Let G be a group and let H and K be normal subgroups of G which intersect trivially. Suppose that $HK = G$. Then $G \cong H \times K$.*

Proof. Define a map $\phi : H \times K \to G$ by $(h, k) \mapsto hk$. Since $G = HK$, this map is surjective.

Let $(h_1, k_1), (h_2, k_2) \in N \times K$. Then

$$\phi((h_1, k_1)(h_2, k_2)) = \phi((h_1h_2, k_1k_2)) = h_1h_2k_1k_2 = h_1k_1h_2k_2 = \phi((h_1, k_1))\phi((h_2, k_2)).$$

Thus ϕ is a homomorphism.

Let $(h, k) \in \ker(\phi)$. Then $hk = 1$. Thus $h = k^{-1}$, so $h \in N \cap K$; thus $h = 1$. Similarly $k = 1$. Thus ϕ is injective. □

Thus we may view G as a direct product of normal subgroups H and K whenever $H \cap K$ is trivial. This is called an *internal direct product*.

4. Semidirect Product

Definition 3. Let G be a group and let $H, K \leq G$. We say the G is the (internal) *semidirect product* of K and H if

- $K \lhd G$;
- $K \cap H = \{1\}$;
- $KH = G$.

In this case we write $G = K \rtimes H$.

Example 1. The dihedral groups are semidirect products. Consider $D_4 = \{\sigma, \tau\}$ where σ is an element of order 4 and τ is an element of order 2. Let $K = \langle \sigma \rangle$ and $H = \langle \tau \rangle$. Then $K \lhd D_4$, $K \cap H = \{\epsilon\}$, and $KH = D_4$.

The simplist nonabelian groups are of the form $K \rtimes H$, where K and H are cyclic. The dihedral groups are exactly the groups of the form $K \rtimes H$, where K is finite cyclic and H has order 2.

Proposition 7. *Let $G = K \rtimes H$. Then $G/K \cong H$.*

Proof. The canonical map $H \to G/K$ given by $h \mapsto hK$ is a homomorphism. Verify that it is bijective. $\square$

Note how the multiplication occurs in semidirect product:

$$k_1 h_1 k_2 h_2 = k_1 (h_1 k_2 h_1^{-1}) h_1 h_2.$$

Let K and H be any groups. Let $\phi : H \to \mathrm{Aut}(K)$. We may construct a semidirect product of K and H relative to ϕ, denoted by $K \rtimes_\phi H$, as follows:

For $h \in H$, denote $\phi(h)$ by ϕ_h. Define a multiplication on the cartesian product $K \times H$ by

$$(k_1, h_1)(k_2, h_2) = (k_1 \phi_{h_1}(k_2), h_1 h_2).$$

Verify that this is associative. The identity is still $(1_K, 1_H)$, and the inverse of (k, h) is $(\phi_{h^{-1}}(k), h^{-1})$.

If $\phi : H \to \mathrm{Aut}(K)$ is the trivial map, then $K \rtimes_\phi H = K \times H$. Otherwise, we get a group which is nonabelian, even if both K and H are abelian. In this way, we may build nonabelian groups out of cyclic groups.

Example 2. Let X denote the graph of $\tan x$ as a subset of $\mathbb{R}^2$, and consider the group of all isometries of X. There are two types: horizontal translation by multiples of π, and rotations by $180°$ about x-intercepts.

Department of Mathematics, BASIS Scottsdale
Email address: paul.bailey@basised.com