I will attempt to respond to some of your questions.

Question 1. Could we transform the coordinate system like what we did in the last chapter to avoid using the theorem from advanced calculus?

Answer. That is very astute; in a manner of speaking, that is what we are doing. □

Question 2. How do we "avoid" the "advanced theorem"?

Answer. We have a function $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^3$. We have a path $\vec{r} : [a, b] \to D$. The velocity is $\vec{v}(t) = \vec{r}'(t)$. The image of the path is a curve C.

We break the image of the path in pieces by partitioning $[a, b]$ with

$$a = t_0 < t_1 < \cdots < t_n = b.$$

We let $\Delta t_k = t_k - t_{k-1}$. We let Δs_k be the *arclength* along the path from $\vec{r}(t_{k-1})$ to $\vec{r}(t_k)$. We pick a point $c_k \in [t_{k-1}, t_k]$, evaluate the function at $\vec{r}(c_k)$, and multiply this by the arclength of that piece. Now the integral we seek is estimated by

$$\sum_{k=1}^{n} f(\vec{r}(c_k))\Delta s_k.$$

The problem is, we haven't computed the arclength. We pull our famous trick:

$$\int_C f\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(\vec{r}(c_k))\Delta s_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(\vec{r}(c_k))\frac{\Delta s_k}{\Delta t_k}\Delta t_k = \int_a^b f(\vec{r}(t))\frac{ds}{dt}\, dt = \int_a^b f(\vec{r}(t))|\vec{v}(t)|\, dt,$$

which works because $|\vec{v}(t)| = \dfrac{ds}{dt}$. It's not terribly formal, but it certainly works. □

Question 3. I understand the material, but wonder how these line integrals will be used.

Answer. In the next section, we will have a vector field defined along the curve. We take the component of the vector field in the direction of the curve by projection; this gives a scalar at each point along the curve. Then we integrate. □

Question 4. In what cases does the integral of a value along a curve joining two points not differ if you change the path between them? Are there any specific functions in which this special case occurs?

Answer. For vector fields, this is the crux of section 16.3. The answer will be that if you can continuously deform one path to the other through the domain of differentiability of the vector field, the integral should remain unchanged. This is a wonderful, beautiful theorem with real world consequences. It will take us a while to understand it. □

Question 5. How do you do Problem 20?

Answer. Oh no I'm almost out of time, I will definitely get to that tomorrow. □