AP Calculus AB — Responses 03/25

Dr. Paul L. Bailey — Tuesday, March 24, 2020

Please do not attempt the material until you have read the section.
Hints for various problems:

- §5.3 # 10: $\ln(a) - \ln(b) = \ln(a/b)$.
- §5.3 # 26: $\ln(ab) = \ln(a) + \ln(b)$.
- §5.3 # 54: Let $u = \dfrac{-1}{x^2}$.
- §5.4 # 15: $\pi$ is an irrational constant, use the power rule.
- §5.4 # 47: $\dfrac{d}{dx} a^x = (\ln a)a^x$, so $\int a^x \, dx = \dfrac{a^x}{\ln a}$.
- §5.4 # 53: Let $u = \cos t$.
- §5.4 # 59: $\sqrt{2}$ is an irrational constant. Use the power rule.
- §5.4 # 87: $\log_{20}(17) = \dfrac{\ln 20}{\ln 17}$.

**Problem 1** (Thomas §7.3 # 10)**.** Find $y$ such that $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$.

*Solution.* The ingredients are:

- Treat $x$ as a constant.
- $\ln(a) - \ln(b) = \ln(b/a)$
- ln is injective, so $\ln a = \ln b$ if and only if $a = b$

So:

$$\begin{aligned}
\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x) &\Leftrightarrow \ln\left(\frac{y^2 - 1}{y + 1}\right) = \ln(\sin x) \\
&\Leftrightarrow \ln(y - 1) = \ln(\sin x) \\
&\Leftrightarrow y - 1 = \sin x \\
&\Leftrightarrow y = \sin x + 1
\end{aligned}$$

So

$$\boxed{y = \sin x + 1.}$$

□

**Problem 2** (Thomas §7.3 # 26)**.** Find $\frac{dy}{dx}$, where $y = \ln(3\theta e^{-\theta})$.

*Solution.* The ingredients are or could be:

- $\frac{d}{dx} \ln x = \frac{1}{x}$
- $\frac{d}{dx} e^x = e^x$ (we won't actually need this)
- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(e^x) = x$
- The chain rule (we don't actually need this either)

So

$$\frac{d}{dx} \ln(3\theta e^{-\theta}) = \frac{d}{dx} \ln(3) + \ln(\theta) + \ln(e^{-\theta}) = \frac{d}{dx} \ln 3 + \frac{d}{dx} \ln(\theta) + \frac{d}{dx}(-\theta) = 0 + \frac{1}{\theta} - 1,$$

that is,

$$\boxed{\frac{dy}{dx} = \frac{1}{\theta} - 1.}$$

□

**Problem 3** (Thomas §7.4 # 9)**.** Find $x$ such that $3^{\log_3(x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10}(2)}$.

*Proof.* Since $a^{\log_a(x)} = x$, and $\log_a(a^x) = x$,

$$3^{\log_3(x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10}(2)} \Leftrightarrow x^2 = 5x - 6 \Leftrightarrow x^2 - 5x + 6 = 0 \Leftrightarrow x = 2, 3.$$

Thus

$$\boxed{x = 2 \text{ or } x = 3.}$$

□

**Problem 4** (Thomas §7.4 # 15)**.** Computer $\frac{d}{dx} x^\pi$.

*Solution.* Since $\pi$ is an irrational constant, we use the power rule for irrational exponents as explain at the bottom of page 492.

$$\boxed{\frac{d}{dx} x^\pi = \pi x^{\pi - 1}.}$$

□

**Problem 5** (Thomas §7.4 # 26)**.** Let $y = \log_{25} e^x - \log_5(\sqrt{(x)})$. Find $\dfrac{dy}{dx}$.

*Solution.* This is probably clearest if we first convert to natural log.

We know the *change of base formula* for logarithms:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

If $a = e$, we have the natural log on the right:

$$\log_b(x) = \frac{\ln x}{\ln b}.$$

Thus

$$y = \log_{25} e^x - \log_5(\sqrt{(x)}) = \frac{\ln e^x}{\ln 25} - \frac{\ln(\sqrt{x})}{\ln 5}.$$

Since $\ln e^x = x$, $\ln 25 = 2\ln 5$, and $\ln(\sqrt{x}) = \dfrac{1}{2}\ln x$, we have

$$y = \frac{x}{2\ln 5} - \frac{x}{2\ln 5} = 0.$$

Thus

$$\frac{dy}{dx} = 0.$$

□

**Problem 6** (Thomas §7.3 # 54)**.** Compute $\displaystyle\int \frac{e^{-1/x^2}}{x^3}\,dx$.

*Answer.* We look around the integrand and try to spot one thing which is the derivative of another thing. I know that $\dfrac{1}{x^3}$ is more or less the derivative of $\dfrac{1}{x^2}$ (give a take multiplication by a constant). So we let $u$ be the thing whose derivative is lying around.

Let $u = \dfrac{-1}{x^2}$ so that $du = \dfrac{2}{x^3}dx$. Now

$$\int \frac{e^{-1/x^2}}{x^3}\,dx = \frac{1}{2}\int e^{-1/x^2}\frac{2}{x^3}\,dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{-1/x^2}.$$

Thus

$$\boxed{\int \frac{e^{-1/x^2}}{x^3}\,dx = \frac{1}{2}e^{-1/x^2} + C.}$$

□

**Question 1.** Can pH, Richter scale, and sound intensity only be calculated using the common logarithm?

*Answer.* These units happen to be computed using base 10 logarithms. We should be aware, however that this is because humans are used to thinking in base 10, not because of any aspect of nature (other than our number of fingers) that revolves around the number 10. □

**Question 2.** How is the learning strategy going to change, if at all, since the new AP format has been announced?

*Answer.* For now, we need to finish learning some outstanding topics. These are exponential growth, slope fields, separable differential equations, and L'Hospital's rule.

Then, we will try to focus on the content and style of the upcoming on-line AP exam, whose nature is yet to be completely determined. □