

**Problem 1** (Thomas §16.2 # 16). Find the work done by  $\vec{F} = \langle 6z, y^2, 12x \rangle$  along the path  $\vec{r}(t) = \langle \sin t, \cos t, t/6 \rangle$ .

*Solution.* The vectors along the path are given by plugging  $\vec{r}$  into  $\vec{F}$ , thusly:

$$\vec{F}(\vec{r}(t)) = \langle t, \cos^2 t, 12 \sin t \rangle.$$

The derivative of  $\vec{r}$  is

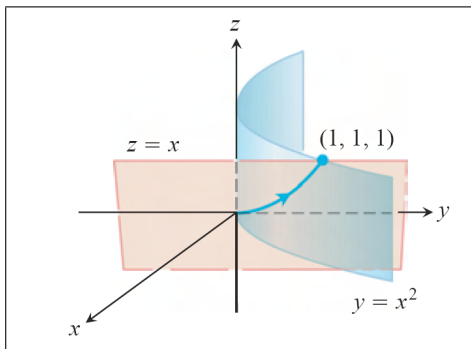
$$\vec{v}(t) = \langle \cos t, -\sin t, \frac{1}{6} \rangle.$$

So

$$\begin{aligned} \text{Work} &= \int_0^{2\pi} \vec{F} \cdot \vec{v} \, dt \\ &= \int_0^{2\pi} \langle t, \cos^2 t, 12 \sin t \rangle \cdot \langle \cos t, -\sin t, \frac{1}{6} \rangle \, dt \\ &= \int_0^{2\pi} t \cos t - \cos^2 t \sin t + 2 \sin t \, dt \\ &= (t \sin t + \cos t) + \left( \frac{1}{3} \cos^3 t \right) - (2 \cos t) \Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

□

**Problem 2** (Thomas §16.2 # 43). The field  $\vec{F}(x, y, z) = \langle xy, y, -yz \rangle$  is the velocity field of a flow in space. Find the flow from  $(0, 0, 0)$  to  $(1, 1, 1)$  along the curve of intersection of the cylinder  $y = x^2$  and the plane  $z = x$ .



*Solution.* Along the curve, we have  $y = x^2$  and  $z = x$ , so the curve is parameterized by

$$\vec{r}(t) = \langle t, t^2, t \rangle,$$

whose derivative is

$$\vec{v}(t) = \langle 1, 2t, 1 \rangle.$$

Along this curve, the flow is

$$\vec{F}(\vec{r}(t)) = \langle t^3, t^2, -t^3 \rangle.$$

Now

$$\begin{aligned} \text{Flow} &= \int_0^1 \vec{F} \cdot \vec{v} \, dt \\ &= \int_0^1 \langle t^3, t^2, -t^3 \rangle \cdot \langle 1, 2t, 1 \rangle \, dt \\ &= \int_0^1 t^3 + 2t^3 - t^3 \, dt \\ &= \int_0^1 2t^3 \, dt \\ &= \left. \frac{t^4}{2} \right|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

□

**Problem 3** (Thomas §16.2 # 44). Find the flow of the field  $\vec{F} = \nabla(xy^2z^3)$  along these paths.

(a) Once around the curve  $C$ , which is the ellipse which is the intersection of the plane  $2x + 3y - z = 0$  and the cylinder  $x^2 + y^2 = 12$ , clockwise as viewed from above.

(b) Along the line segment from  $(1, 1, 1)$  to  $(2, 1, -1)$ .

*Attempted Solutions.* I started to do these problems based on what was clear just from Section 16.2, and I will show you how far I got before I looked for an easier way. This will give an example of why theorems are useful.

(a) Let  $\alpha = \sqrt{12}$ .

Along  $C$ , we have  $z = 2x + 3y$ , so the curve is the image of the path

$$\vec{r}(t) = \langle \alpha \cos t, \alpha \sin t, 2\alpha \cos t + 3\alpha \sin t \rangle.$$

Its derivative is

$$\vec{v}(t) = \langle -\alpha \sin t, \alpha \cos t, -2\alpha \sin t + 3\alpha \cos t \rangle.$$

The gradient is

$$\vec{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle.$$

When we attempt to plug this into Along the path, this is

$$\vec{F}(\vec{r}(t)) = \dots$$

I didn't really want to plug this in.

(b) The line segment from point  $A$  to point  $B$  is parameterized as  $A + t(B - A)$  for  $t \in [0, 1]$ .

In our case,

$$\vec{r}(t) = (1, 1, 1) + t\langle 1, 0, -2 \rangle = \langle 1 + t, 1, 1 - 2t \rangle,$$

whose derivative is

$$\vec{v}(t) = \langle 1, 0, -2 \rangle.$$

The gradient is

$$\vec{F}(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle.$$

Along the path, this is

$$\vec{F}(\vec{r}(t)) = \langle (1 - 2t)^3, 2(1 + t)(1 - 2t)^3, 3(1 + t)(1 - 2t)^3 \rangle.$$

So

$$\begin{aligned} \text{Flow} &= \int_0^1 \vec{F} \cdot \vec{v} \, dt \\ &= \int_0^1 \langle (1 - 2t)^3, 2(1 + t)(1 - 2t)^3, 3(1 + t)(1 - 2t)^3 \rangle \cdot \langle 1, 0, -2 \rangle \, dt \\ &= \int_0^1 (1 - 2t)^3 - 6(1 + t)(1 - 2t)^3 \, dt \\ &= \dots \end{aligned}$$

At this point, I didn't what to multiply out this quartic polynomial. I could have, but ...

□

**Lemma 1.** Let  $D \subset \mathbb{R}^n$  and let  $I \subset \mathbb{R}$  be an interval. Let  $f : D \rightarrow \mathbb{R}$  and let  $\vec{F} : D \rightarrow \mathbb{R}^n$  be given by  $\vec{F} = \nabla f$ . Let  $\vec{r} : I \rightarrow D$  be a path in  $D$ . Then

$$\nabla f \cdot \frac{d\vec{r}}{dt} = \frac{d}{dt} f(\vec{r}(t)).$$

*Proof.* This is the chain rule:

$$\frac{d}{dt} f(\vec{r}(t)) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt}.$$

□

**Lemma 2.** Let  $D \subset \mathbb{R}^n$  and let  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $f : D \rightarrow \mathbb{R}$  and let  $\vec{F} : D \rightarrow \mathbb{R}^n$  be given by  $\vec{F} = \nabla f$ . Let  $\vec{r} : [a, b] \rightarrow D$  be a path in  $D$ . Then the flow of  $\vec{F}$  along  $\vec{r}$  is given by

$$\int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

*Proof.* From the previous lemma and the Fundamental Theorem of Calculus,

$$\text{Flow} = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_a^b \frac{d}{dt} f(\vec{r}(t)) dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

□

**Problem 3** (Thomas §16.2 # 43 - Second Attempt). The field  $\vec{F}(x, y, z) = \langle xy, y, -yz \rangle$  is the velocity field of a flow in space. Find the flow from  $(0, 0, 0)$  to  $(1, 1, 1)$  along the curve of intersection of the cylinder  $y = x^2$  and the plane  $z = x$ .

*Solution* Thomas §16.2 # 44. Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be given by  $f(x, y, z) = xy^2z^3$ , so that  $\vec{F} = \nabla f$ . If we use the lemma, we don't actually need to compute the gradient, just the endpoints of the domain of the curve.

(a) Along  $C$ , we have  $z = 2x + 3y$ , so the curve is the image of the path

$$\vec{r} : [0, 2\pi] \rightarrow \mathbb{R}^3 \quad \text{given by} \quad \vec{r}(t) = \langle \alpha \cos t, \alpha \sin t, 2\alpha \cos t + 3\alpha \sin t \rangle.$$

So in this case,  $a = 0$  and  $b = 2\pi$ .

Note that the ellipse is a closed loop, so  $\vec{r}(0) = \vec{r}(2\pi) = (\alpha, 0, 2\alpha)$ ; that is,  $\vec{r}(b) = \vec{r}(a)$ , so

$$\boxed{\text{Flow} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0.}$$

(b) In this case,

$$\vec{r} : [0, 1] \rightarrow \mathbb{R}^3 \quad \text{is given by} \quad \vec{r}(t) = (1, 1, 1) + t\langle 1, 0, -2 \rangle = \langle 1 + t, 1, 1 - 2t \rangle,$$

so  $a = 0$ ,  $b = 1$ ,  $\vec{r}(a) = (1, 1, 1)$ , and  $\vec{r}(b) = (2, 1, -1)$ , so

$$\text{Flow} = f(\vec{r}(b)) - f(\vec{r}(a)) = -2 - 1 = -3.$$

□