AP Calculus AB — Responses 0330
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Problem 1. Use the fact that exp and log are injective to solve these problems. That is, if $\exp a = \exp b$, then $a = b$, and if $\log a = \log b$, then $a = b$.

(a) Find all $x \in \mathbb{R}$ such that $7^{x^2-4x+1} = 49^{x-2}$.

Solution. We use the fact that $(a^b)^c = a^{bc}$, and that exponential functions are injective, which is to say that if f is an exponential function, then $f(a) = f(b) \Rightarrow a = b$.

$$\begin{aligned} 7^{x^2-4x+1} = 49^{x-2} &\Leftrightarrow 7^{x^2-4x+1} = 7^{2(x-2)} \\ &\Leftrightarrow x^2 - 4x + 1 = 2(x-2) \\ &\Leftrightarrow x^2 - 6x + 5 = 0 \\ &\Leftrightarrow (x-2)(x-3) = 0 \\ &\Leftrightarrow x = 2 \text{ or } x = 3. \end{aligned}$$

□

(b) Find all $x \in \mathbb{R}$ such that $\ln(x+1) + \ln(x+2) = \ln(x+3)$.

Solution. Here we use that $\log(a) + \log(b) = \log(ab)$, and that logarithmic functions are injective.

$$\begin{aligned} \ln(x+1) + \ln(x+2) = \ln(x+3) &\Leftrightarrow \ln((x+1)(x+2)) = \ln(x+3) \\ &\Leftrightarrow (x+1)(x+2) = x+3 \\ &\Leftrightarrow x^2 + 3x + 2 = x + 3 \\ &\Leftrightarrow x^2 + 2x - 1 = 0 \\ &\Leftrightarrow x = \frac{-2 \pm \sqrt{4+4}}{2} \\ &\Leftrightarrow x = -1 \pm \sqrt{2} \end{aligned}$$

However, this is only true for those x which are in the domain of the equation. We see that if $x \leq -1$, we cannot plug it into $\log(x+1)$. So, only $x = \sqrt{2} - 1$ is a correct solution. □

Problem 2. Solve the initial value problem

$$\frac{dy}{dx} = 3x^2 - 4 \qquad \text{where } y(2) = 5.$$

Solution. First we integrate:

$$\frac{dy}{dx} = 3x^2 - 4 \Rightarrow y = \int 3x^2 - 4 \Rightarrow y = x^3 - 4x + C.$$

Now we need to find the C that works for the initial condition $y(2) = 5$. We have $5 = y(2) = 2^3 - 4 \cdot 2 + C = C$, so $C = 5$. Thus □

Problem 3. Let f and g be functions that are differentiable everywhere, such that g is the inverse function of f. Suppose that $g(-2) = 5$ and $f'(5) = -\frac{1}{2}$. Find $g'(-2)$.

Solution. If f is differentiable at $x = a$ and $f'(a) \neq 0$, then f has an inverse function g is an open set around a. This is because if $f'(a)$ is positive, then f is increasing around a, so f is injective around a; if $f'(a)$ is negative, then f is decreasing, hence injective, hence invertible, near a.

If $f(a) = b$, then $g(b) = a$. The slope of the line tangent to the graph of g at (b, a) is the reciprocal of the slope of the line tangent to the graph of f at (a, b). Therefore,

$$g'(b) = \frac{1}{f'(a)}.$$

In our case, $a = 5$ and $b = -2$. So

$$g'(-2) = \frac{1}{f'(5)} = \frac{1}{(-1/2)} = -2.$$

□

Problem 4. Compute

$$\int \frac{\sec^2 x \, dx}{\sqrt{1 - \tan^2 x}}.$$

Solution. Let $u = \tan x$ so that $du = \sec^2 x \, dx$. Then

$$\int \frac{\sec^2 x \, dx}{\sqrt{1 - \tan^2 x}} = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin(u) + C = \arcsin(\tan x) + C.$$

□