Vector Calculus	Responses 0330
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I think this is all correct, please let me know if you find an error.

Problem 1. When evaluating a path integral, you are always asked to make sure the parametrization for the path is smooth. What does that mean and why is that a necessary condition before integrating?

Answer. According to Thomas, a path is *smooth* if it is differentiable, with continuous and nonvanishing derivatives. If this is the case, the path has a well-defined nonzero tangent vector at every point.

We will typically be using the formula

$$\int_C f(x,y) \, ds = \int_a^b f(\vec{r}(t)) |\vec{v}(t)| \, dt,$$

where $\vec{v}(t) = \frac{d\vec{r}}{dt}$. So, $\vec{v}(t)$ must be at least piecewise differentiable to evaluate this integral. Moreover, work and flow integral require a vector field to be dotted with the velocity vector, so that velocity vector better exist.

Problem 2. Consider the vector fields in \mathbb{R}^2 given by

$$ec{F}(x,y) = \langle x,y
angle \quad ext{and} \quad ec{G}(x,y) = \langle -y,x
angle.$$

Let C denote the unit circle, oriented counterclockwise.

- (a) Draw a small sketch of each of these vector fields.
- (b) Compare the flow of \vec{F} versus \vec{G} along C.
- (c) Compare the flux of \vec{F} versus \vec{G} along C.

Write clearly, in complete sentences. Explain your reasoning.

Solution. The vector field \vec{F} has vector that are point away from the origin, which get longer the farther away from the origin it gets. The vector field \vec{G} has vectors that are perpendicular to the position vector at a point, so they look like they are going around the origin, and get longer the farther away from he origin they get.

The flow of \vec{F} along the unit circle is zero, since the field vectors are perpendicular to the tangent vectors, so the dot products are all zero. The flow of \vec{G} is maximal, since the field vectors are parallel to the tangent vectors. To compute this, parameterize the curve:

$$\vec{r}: [0, 2\pi] \to \mathbb{R}^2$$
 given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

Then the velocity is

$$\vec{v}(t) = \langle -\sin t, \cos t \rangle.$$

Now

$$\vec{G}(\vec{r}(t)) = \langle -\sin t, \cos t \rangle$$

 \mathbf{SO}

$$Flow(G) = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t = 2\pi.$$

The flux of \vec{G} along the unit circle is zero, again because the vectors are perpendicular, but the flux is maximal, since the vectors are parallel, and the flux is computed to be \vec{F} is 2π .

Problem 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous real valued function defined on \mathbb{R}^2 .

Let C be a smooth curve in \mathbb{R}^2 parameterized by $\vec{r} : [0,1] \to \mathbb{R}^2$ with parameter t. Compare and contrast the integrals below numerically (what "answer" do you get?), geometrically, and in words.

$$\int_C f(x,y) \, ds \quad \text{versus} \quad \int_0^1 f(\vec{r}(t)) \, dt$$

Solution. These are different, unless the curve is parameterized by arclength, so the speed is one. This is because

$$\int_C f(x,y) \, ds = \int_0^1 f(\vec{r}(t)) |\vec{v}(t)| \, dt \neq \int_0^1 f(\vec{r}(t)) \, dt.$$

What does this mean "geometrically"? The line integral $\int_C f(x, y) ds$ is independent of how the curve is parameterized. Any smooth parameterization \vec{r} of the curve can be computed as $\int_C f(x, y) ds = \int_0^1 f(\vec{r}(t)) |\vec{v}(t)| dt$.

However, the value of the integral $\int_0^1 f(\vec{r}(t)) \, does$ depend on the parameterization. If the particle goes faster, but the factor of $|\vec{v}|$ is not incorporated into the integrand, the resulting value will actually be less than the line integral.

Problem 4. Let $\vec{F} = \langle -y, x \rangle$. Let C_1 denote the semicircle of radius 1 in the upper half plane from (1,0) to (-1,0) Let C_2 denote the line segment along the x-axis from (1,0) to (-1,0).

- (a) Compute the flow of \vec{F} along C_1 .
- (b) Compute the flow of \vec{F} along C_2 .
- (c) Compare these values. Are they the same? Is \vec{F} conservative?

Solution. The curve C_1 is parameterized by

$$\vec{r}_1(t): [0,\pi] \to \mathbb{R}^2 \quad \text{by} \quad \vec{r}(t) = \langle \cos t, \sin t \rangle,$$

 \mathbf{SO}

$$\vec{v}_1(t) = \langle -\sin t, \cos t \rangle.$$

We have

$$\operatorname{Flow}_{C_1} = \int_0^{\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{\pi} \, dt = \pi.$$

The curve C_2 is parameterized by

$$\vec{r}_2: [0,1] \to \mathbb{R}^2$$
 by $\vec{r}(t) = (1,0) + t \langle -2, 0 \rangle$

 \mathbf{so}

$$\vec{v}_2(t) = \langle -2, 0 \rangle$$

Thus

Flow_{C2} =
$$\int_0^1 \langle 0, 1 - 2t \rangle \cdot \langle -2, 0 \rangle dt = 0.$$

The field is not conservative, because it does not admit path independence.