Vector Calculus Responses 0331
Dr. Paul L. Bailey Monday, March 30, 2020

Question 1. Could you explain how f is derived in Example 2 on page 1165?

Answer. Okay I'll try. First let me convert to my preferred notation.

Let $\vec{F}(x,y,z) = \langle e^x \cos y + yz, xz - e^x \sin y, xy + z \rangle$. Our goal is to find a scalar function $f(x,y,z)$ whose gradient is $\vec{F}$.

Since $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$, we set each of these partials to the corresponding component of $\vec{F}$:n

$$\frac{\partial f}{\partial x} = e^x \cos y,\ \frac{\partial f}{\partial y} = xz - e^x \sin y,\ \text{and}\ \frac{\partial f}{\partial z} = xy + z.$$

At this point, the book integrates the first of these with respect to x to get

$$f = \int \frac{\partial f}{\partial x} dx = e^x \cos y + xyz + g(y,z).$$

Since y and z are treated as constants when taking the partial with respect to x, the constant of integration must be treated as a function of y and z. They called that constant $g(y,z)$.

But we want $\frac{\partial f}{\partial y}$ to equal the second component of $\vec{F}$; that is, we want $\frac{\partial f}{\partial y} = xz - e^x \sin y$, so we now take the partial of f with respect to y and get

$$\frac{\partial f}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^z \sin y.$$

So, $\frac{\partial g}{\partial y}$ is zero, whence g is a function only of z:

$$f = e^x \cos y + xyz + g(z).$$

Finally, take the partial of this with respect to z, and set that equal to the third component of $\vec{F}$:

$$\frac{\partial f}{\partial z} = xy + \frac{dg}{dz} = xy + z;$$

thus $\frac{dg}{dz} = z$, so $g = \frac{z^2}{2} + C$ for some constant C. Putting these things together gives:

$$f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

I feel like I just wrote what the book wrote! But maybe it helped. □

Question 2. For the component test for conservative fields, why are the conditions for the partials so particular (why does partial P/partial y have to equal partial N/ partial z and not something else)?

Answer. Notice that the hypothesis for this theorem insists that the vector field $\vec{F}$ has continuous first partial derivatives. So, if it is the gradient of f, the second partials of f commute; that is, $\frac{\partial^2 f}{dz\,dy} = \frac{\partial^2 f}{dy\,dz}$. This is the gist of the proof given in the book that if $\vec{F}$ has a potential, then these partial equations hold. The converse, that if these equations hold, then $\vec{F}$ has a potential, is much more difficult and is omitted until a later section. □

Question 3. Do #3 and #4 on the practice quiz? I think I might've messed the computations up...

Let's do 3.

Problem 1. Let $\vec{F}(x,y) = \langle xy^2, x^2y\rangle$. Let $C = C_1 \cup C_2$, where

$$C_1 : \vec{r}_1 : [0,\pi] \to \mathbb{R}^2 \quad \text{given by} \quad \vec{r}_1(t) = \langle 6\cos t, 6\sin t\rangle,$$

and

$$C_2 : \vec{r}_2 : [-6,6] \to \mathbb{R}^2 \quad \text{given by} \quad \vec{r}_2(t) = \langle t, 0\rangle.$$

Compute the circulation of $\vec{F}$ along C.

Solution. We compute these independently:

$$\begin{aligned}
\int_{C_1} \vec{F}\cdot d\vec{r} &= \int_0^{\pi} \langle xy^2, x^2y\rangle \cdot \langle -6\sin t, 6\cos t\rangle\, dt \\
&= \int_0^{\pi} \langle 6^3 \cos t \sin^2 t, 6^3 \sin t \cos^2 t\rangle \cdot \langle -6\sin t, 6\cos t\rangle\, dt \\
&= \int_0^{\pi} 6^4 \sin^3 t \cos t + 6^4 \cos^3 t \sin t\, dt \\
&= 6^4 \left[\frac{\sin^4 t}{4} - \frac{\cos^4 t}{4}\right]_0^{\pi} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\int_{C_2} \vec{F}\cdot d\vec{r} &= \int_{-6}^{6} \langle xy^2, x^2y\rangle \cdot \langle 1, 0\rangle\, dt \quad \text{(along the x-axis, y is zero.)} \\
&= \int_{-6}^{6} \langle 0, 0\rangle \cdot \langle 1, 0\rangle\, dt \\
&= 0.
\end{aligned}$$

So the circulation is 0. We could have guessed this; the vector field is conservative with potential $\frac{x^2y^2}{2}$. □