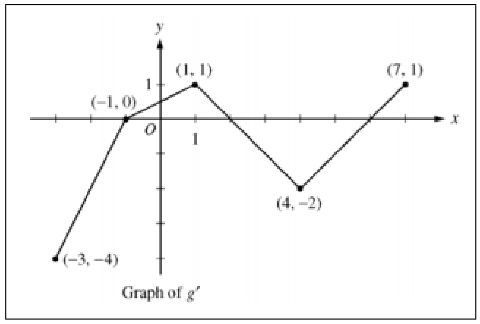
AP Calculus AB — Responses 0407

Dr. Paul L. Bailey — Monday, April 6, 2020

Problem 1. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown below for $-3 \le x \le 7$.



(A) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

Solution. A point of inflection occurs where the concavity changes, which occurs when the first derivative changes direction (increasing to decreasing or vice versa). This occurs at $x = 1$ and at $x = 4$. □

(B) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

Solution. An absolute maximum occurs either at a local maximum or at an endpoint. A local maximum occurs if the derivative changes from positive to negative; the only local maximum is at $x = 2$. So we evaluate g at $x = 2$ and at the endpoints $x = -3$ and $x = 7$. We compute the areas of triangle to find the integrals below.

- $g(2) = 5$
- $g(7) = g(2) + \int_2^7 g'(x)\,dx = 5 - 4 + \dfrac{1}{2} = \dfrac{3}{2}$
- $g(-3) = g(2) + \int_2^{-3} g'(x)\,dx = 5 - \dfrac{3}{2} + 4 = \dfrac{15}{2}$

So, the absolute maximum value is $\dfrac{15}{2}$. □

(C) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

Solution. The average rate of change of $g(x)$ is

$$\frac{g(7) - g(-3)}{7 - (-3)} = \frac{\frac{3}{2} - \frac{15}{2}}{10} = -\frac{3}{5}.$$

□

(D) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

Solution. The average rate of change of $g'(x)$ is

$$\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{10} = \frac{1}{2}.$$

The Mean Value Theorem does not apply to g' on the interval $[-3, 7]$, since g' is not differentiable on $[-3, 7]$. □

Problem 2. Consider the differential equation $dy/dx = e^y(3x^2 - 6x)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(A) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

Solution. We plug $x = 1$ and $y = 0$ into the differential equation to find that the slope of the line is

$$\frac{dy}{dx} = e^0(3 - 6) = -3.$$

Thus the equation for the line is

$$y = -3(x - 1) + 0, \text{ so } f(1.2) \approx -3(1.2 - 1) = -0.6.$$

□

(B) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

Solution. First we separate the variables and slap some integral signs in front to get

$$\int \frac{dy}{e^y} = \int (3x^2 - 6x)\, dx.$$

Then we take antiderivatives to get

$$-e^{-y} = x^3 - 3x^2 + C.$$

Plug in $(1, 0)$ and solve for C to get

$$-e^0 = 1 - 3 + C, \quad \text{so} \quad C = 1.$$

Solve for y and plug in C to get

$$y = -\ln(-(x^3 - 3x^2 + 1)).$$

□

Question 1. For Problem 10 (#26) on the test solutions, can L'Hospital's Rule be applied to every ln or log?

Answer. L'Hospital's Rule is used to compute limits of functions that can be expressed as a fraction. If f and g are differentiable at $x = a$, *and* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

We call this the 0/0 form. The rule can be modified to work for the so-called indeterminate forms ∞/∞, $\infty \cdot 0$, and $\infty - \infty$, as described in Thomas, page 296.

Now log is differentiable at $x = a$ as long as $a > 0$, so you can use L'Hospital's rule to find limits at $a > 0$ of functions that can be put in one of these indeterminate forms. □

Question 2. Which units are not on the AP test? How do you interpret whether or not a second derivative is positive, negative, or 0 from the first derivative of a function?

Answer. The only units not included in this year's AP exam are areas, volumes, and linear motion problems requiring integration.

The second derivative is the derivative of the derivative. So,

- $f''(x)$ is positive if $f'(x)$ is increasing.
- $f''(x)$ is negative if $f'(x)$ is decreasing.
- $f''(x) = 0$ if f' changes sign at x.

□

Question 3. When you are presented a graph of the derivative function (g'(x)), like the one on today's AP Classroom, how can we figure out information about the original function (g(x))?

Answer. This is an important question because the AP loves to ask this sort of thing.

Similarly to above,

- g is increasing where g' is positive (above the x-axis).
- g is decreasing where g' is negative (below the x-axis).
- g has a local extremum where g' crosses the x-axis.
- g has a local maximum at x if g' changes from positive to negative at x.
- g has a local minimum at x if g' changes from negative to positive at x.

Please think about why these things are true; it will help you to remember them.

Mrs. Bailey has made a nice review sheet on this topic, I will share it with you. □

Question 4. How can we format our math digitally for the AP?

Proof. This is a good question.

1) You can fake it. For example: `int x^3 + 7x dx = x^4/4 + 7/3 x^3 + C`
2) You can write on paper and scan it.
3) They might have some sort of equation editor in the exam, they don't seem to have it yet.
4) The best, of course, is AMS Latex, but that takes a lot of time to learn.

I presume that AP will give us guidance on this eventually. □