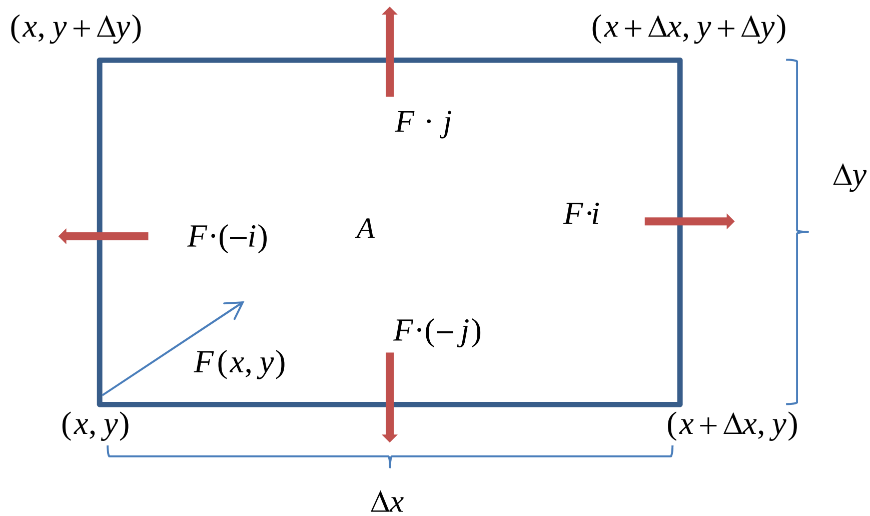
Divergence:

Let $F(x,y)=(f_1(x,y), f_2(x,y))$ be the velocity field of a fluid flowing in the plane, and that the first partial derivatives of f_1 and f_2 are continuous at each point in a region *R*. Suppose *A* is a small rectangle with one corner at (x,y), whose sides are parallel to the coordinate axes, and such that $A \subset R$. Also, suppose that the side lengths are Δx and Δy.



Notice that the vector field is dotted with the vector in the direction of the red arrows to denote the rate at which the fluid is leaving the rectangle in that direction. To approximate the flow rate at the point (*x*, *y*), we calculate the approximate flow rates across each edge in the directions of the red arrows, add these rates up, then divide the sum by the area of *A*. To get the actual area, we allow $\Delta x \to 0$ and $\Delta y \to 0$.

The fluid flow rate across the

Top	$f_2(x, y+\Delta y)\Delta x$
Bottom	$-f_2(x,y)\Delta x$
Right	$f_1(x+\Delta x, y)\Delta y$
Left	$-f_1(x,y)\Delta y$

When you add the top and bottom flow rates, you get

$$[f_2(x,y+\Delta y)-f_2(x,y)]\Delta x \approx \frac{\partial f_2}{\partial y}\Delta y \Delta x$$

When you add the right and left flow rates, you get

$$[f_1(x+\Delta x,y)-f_1(x,y)]\Delta y \approx \frac{\partial f_1}{\partial x}\Delta x \Delta y$$

The approximations are due to the continuity of the partial derivatives of f_1 and f_2. Therefore, the flux across the boundary is given by $(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y})\Delta x \Delta y$. Since we still have to divide by the area, we get

$$\text{div } \boldsymbol{F} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}$$

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div}\,\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \qquad (1)$$

The figures below represent the velocity field of a gas flowing in the *xy*-plane. What is the divergence of each vector field and what is its physical meaning?

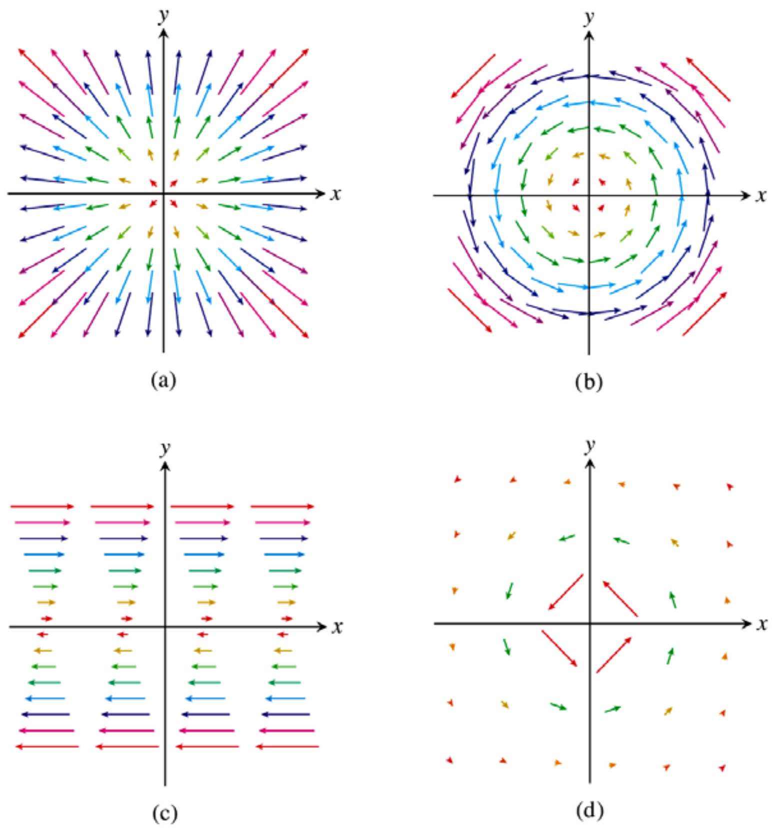


FIGURE 16.28 Velocity fields of a gas flowing in the plane (Example 1).

(a) Uniform expansion or compression: $F(x, y) = (cx, cy)$

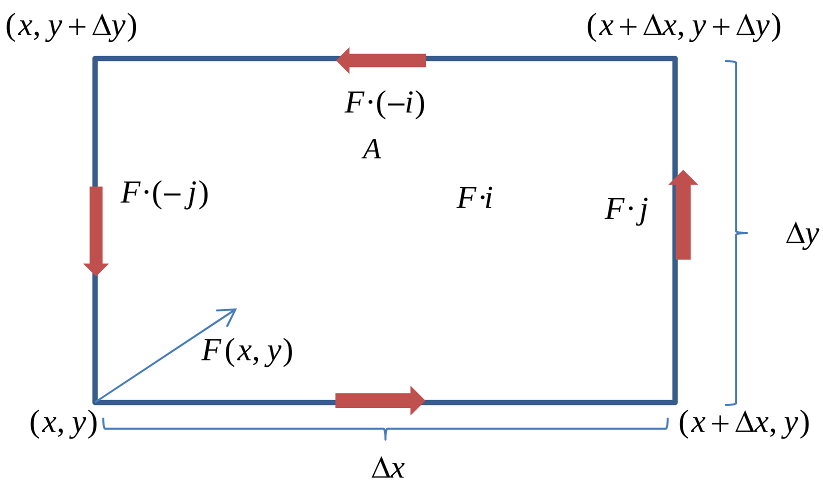
(b) Uniform rotation: $F(x, y) = (-cy, cx)$

(c) Shearing Flow: $F(x, y) = (y, 0)$

(d) Whirlpool Effect: $F(x, y) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$

k-th component of Curl and Circulation Density

" This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region." (Thomas, p. 934) This is called the circulation density, and we will later find out that it is the third coordinate function of a more general concept we call, the *curl*.



We use the exact same concepts as we did to compute the divergence.

Top	$-f_1(x, y+\Delta y)\Delta x$
Bottom	$f_1(x,y)\Delta x$
Right	$f_2(x+\Delta x, y)\Delta y$
Left	$-f_2(x,y)\Delta y$

When you add the top and bottom flow rates, you get

$$[-f_1(x,y+\Delta y)+f_1(x,y)]\Delta x \approx -\frac{\partial f_1}{\partial y}\Delta y\Delta x$$

When you add the right and left flow rates, you get

$$[f_2(x+\Delta x,y)-f_2(x,y)]\Delta y \approx \frac{\partial f_2}{\partial x}\Delta x\Delta y$$

The approximations are due to the continuity of the partial derivatives of f_1 and f_2. Therefore, the flux across the boundary is given by $(-\frac{\partial f_1}{\partial y}+\frac{\partial f_2}{\partial x})\Delta x\Delta y$. Since we still have to divide by the area, we get

$$\text{curl } \boldsymbol{F} = \frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}$$

DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{2}$$

This expression is also called **the k-component of the curl,** denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.

The figures below represent the velocity field of a gas flowing in the *xy*-plane. What is the curl of each vector field and what is its physical meaning?

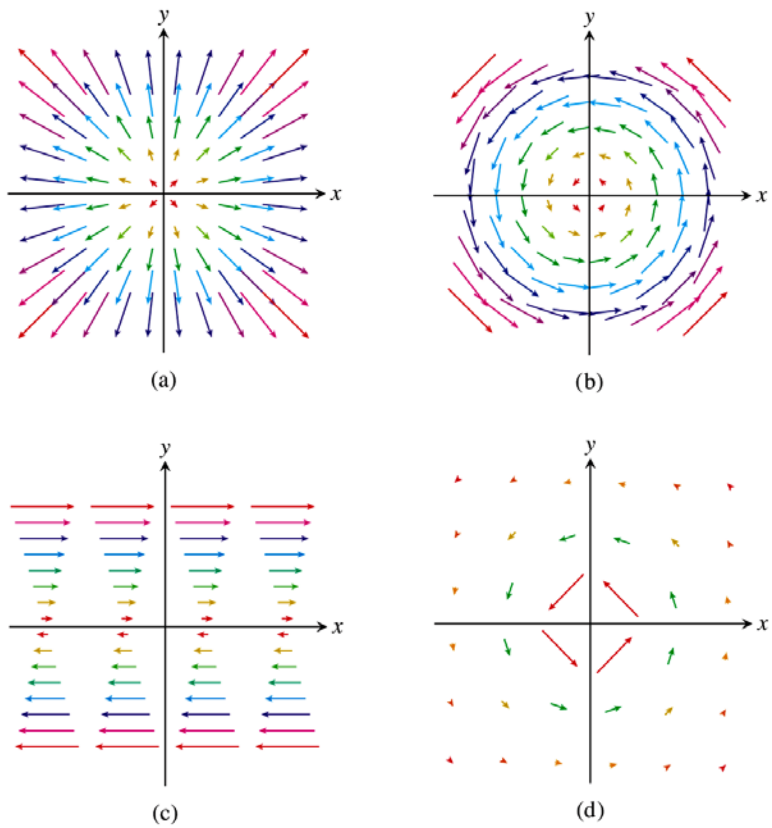


FIGURE 16.28 Velocity fields of a gas flowing in the plane (Example 1).

(a) Uniform expansion or compression: $F(x, y) = (cx, cy)$

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(d) Whirlpool Effect: $F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$

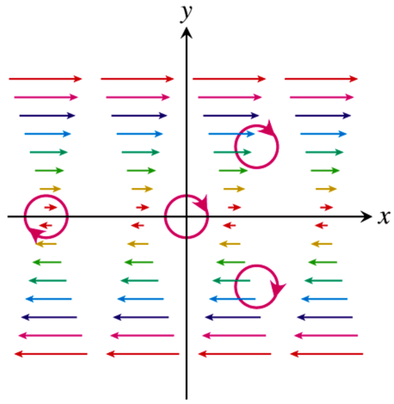
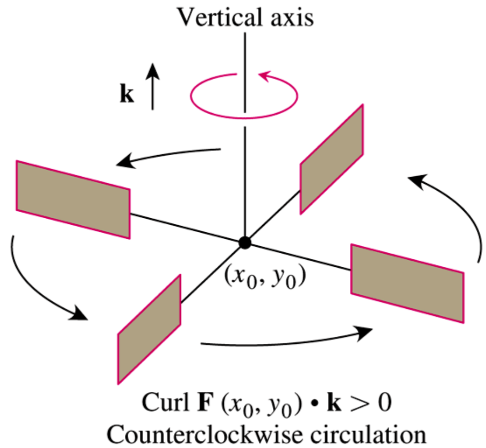


FIGURE 16.31 A shearing flow pushes the fluid clockwise around each point (Example 2c).



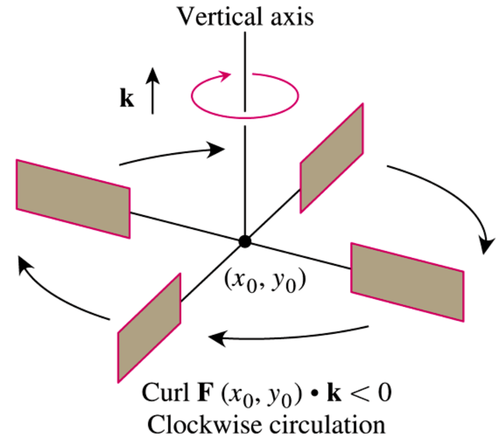


FIGURE 16.30 In the flow of an incompressible fluid over a plane region, the **k**-component of the curl measures the rate of the fluid's rotation at a point. The **k**-component of the curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.

"The *k*-th component of the curl at a point, *P*, gives a way to measure how fast and in what direction a small paddle wheel spins if it is put into the water at the point *P* with its axis perpendicular to the plane and parallel to (0,0,1)." (Thomas, p. 935)

Green's Theorem

Recall that in section 16.2, we defined the flow of a vector field , *F*, along a curve, *C*, and the outward pointing flux of a vector field, *F*, across a curve, *C*, with normal vector *n* is

$$\text{Flux across } C \text{ of } F = \oint_C F \cdot n\, ds$$

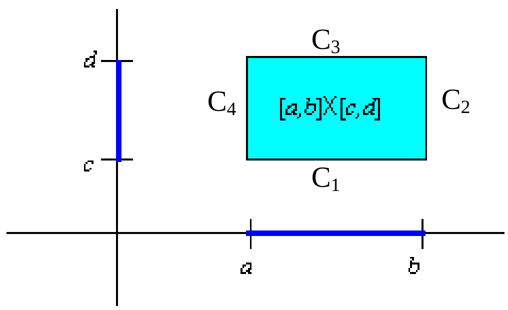
Also, recall that we computed a differential form of the flux of *F* across a simple closed curve, *C*, given by

$$\text{Flux across } C \text{ of } F = \oint_C f_1 \cdot dy - f_2 dx$$

Green's Theorem relates the flux of a vector field across a simple closed curve with the divergence of the vector field, and the flow of a vector field along a simple closed curve with the circulation density of the vector field.

To see the relationship, let $R = \{(x,y) \mid x \in [a,b] \text{ , } y \in [c,d]\}$ be the rectangle with boundary paths, $C_{i.}$

Let *C* be the concatenation of all the curves, C_i, which yields a clockwise boundary path for *R*.



and $F(x,y) = (f_1(x,y).f_2(x,y))$. Then the integral $\iint_R \frac{\partial f_1}{\partial x} dxdy = \int_c^d \int_a^b \frac{\partial f_1}{\partial x} dxdy$ can be easily computed.

$$\begin{aligned}
\int_c^d \int_a^b \frac{\partial f_1}{\partial x} dxdy &= \int_c^d f_1(x,y)\big|_a^b \, dy \\
&= \int_c^d f_1(b,y) - f_1(a,y) dy \\
&= \int_c^d f_1(b,y) dy + \int_d^c f_1(a,y) dy \\
&= \int_{C_2} f_1 dy + \int_{C_4} f_1 dy
\end{aligned}$$

Note that *y* is constant over the curves C_3 and C_1, then $\int_{C_3} f_1 dy + \int_{C_1} f_1 dy = 0$. Therefore,

$$\iint_R \frac{\partial f_1}{\partial x} dxdy = \int_{C_1} f_1 dy + \int_{C_2} f_1 dy + \int_{C_3} f_1 dy + \int_{C_4} f_1 dy = \oint_C f_1 dy$$

A similar method can be used to show that

$$\iint_R \frac{\partial f_2}{\partial y} dxdy = -\oint_C f_2 dx$$

Greens Theorem for Flux:

$$\int \int_R \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} dxdy = \int_{\partial R} f_1 dy - f_2 dx$$

Exercise: Prove the following other version of Green's Theorem.

THEOREM 5—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \qquad (4)$$

Counterclockwise circulation   Curl integral

Proof: