AP Calculus AB — Dr. Paul L. Bailey

Homework 0408 Solutions — Wednesday, April 8, 2020

Problem 1. Let f be the function that is defined for all real numbers x and that has the following properties:

(i) $f''(x) = 24x - 18$

(ii) $f'(1) = -6$

(iii) $f(2) = 0$

(b) Write an expression for $f(x)$.

Solution. Why was this part **(b)**? I'm not sure, but let's do this first. First integrate f'':

$$f'(x) = \int f''(x)\,dx = \int 24x - 18\,dx = 12x^2 - 18x + C.$$

Now find C:

$$-6 = f'(1) = 12x^2 - 18x + C = 12 - 18 + C = -6 + C, \text{ so } C = 0, \text{ so } f'(x) = 12x^2 - 18x.$$

Next integrate f':

$$f(x) = \int f'(x)\,dx = \int 12x^2 - 18x\,dx = 4x^3 - 9x^2 + D.$$

Now find this constant D:

$$0 = f(2) = 4(2)^3 - 9(2)^2 + D = 32 - 36 + D = -4 + D.$$

Thus $D = 4$, and

$$\boxed{f(x) = 4x^3 - 9x^2 + 4.}$$

□

(a) Find each x such that the line tangent to the graph of f at $(x, f(x))$ is horizontal.

Solution. We wish to solve $f'(x) = 0$; that is,

$$12x^2 - 18x = 0, \quad \text{so} \quad 6x(2x-3) = 0, \quad \text{so} \quad \boxed{x = 0 \text{ or } x = \frac{3}{2}.}$$

□

(c) Find the average value of f on the interval $1 \le x \le 3$.

Solution. This is

$$\text{AvgVal} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\int_1^3 4x^3 - 9x^2 + 4\,dx = \frac{1}{2}\Big[x^4 - 3x^3 + 4x\Big]_1^3$$
$$= \frac{1}{2}[(81 - 81 + 12) - (1 - 3 + 4)] = \frac{12-2}{2} = \boxed{5.}$$

□

Problem 2. Let f be the function given by $f(x) = \dfrac{|x| - 2}{x - 2}$.

(a) Find all the zeros of f.

Solution. If possible, we would like to understand the graph of f. This is where much of the information is "stored", or at least, can be revealed through examination of the graph.

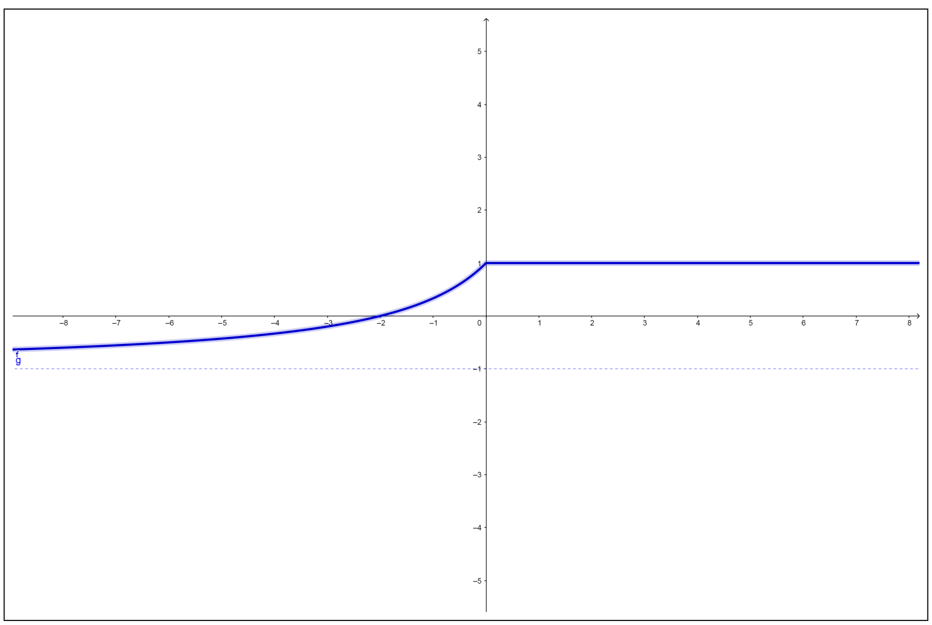
We sketch the graph of f. To do this, break f into cases; if $x \geq 0$, the $|x| = x$, so $f(x) = \dfrac{x-2}{x-2} = 1$. On the other hand, if $x < 0$, then

$$f(x) = \frac{-x-2}{x-2} = -\frac{x+2}{x-2} = -\frac{x-2+4}{x-2} = -1 - \frac{4}{x-2}.$$

Thus

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 - \dfrac{4}{x-2} & \text{if } x < 0 \end{cases}$$

The graph of f is a horizontal line at $y = 1$ for $x \geq 0$. If $x < 0$, the graph of f is the graph of $-\dfrac{4}{x}$ shifted right by 2 and down by 1. So, it has horizontal asymptote at $x = -1$. The graph looks like this:



We see that f has exactly one zero, when $x < 0$, so we solve

$$-1 - \frac{4}{x-2} = 0 \quad \Rightarrow \quad -4 = x - 2 \quad \Rightarrow \quad \boxed{x = -2.}$$

□

(b) Find $f'(1)$.

Solution. To compute the derivative of a piecewise defined function, just take the derivative of each piece.

$$f'(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ \dfrac{4}{(x-2)^2} & \text{if } x < 0 \end{cases}$$

Now plug in:

$$f'(1) = \boxed{0.}$$

□

(c) Find $f'(-1)$.

Solution. We have

$$f'(-1) = \frac{4}{(-1-2)^2} = \boxed{\frac{4}{9}.}$$

□

(d) Find the range of f. We read this off from the graph of f; it is the set of all y such that $f(x) = y$ for some x. This is

$$\boxed{(-1, 1] = \{x \in \mathbb{R} \mid -1 < x \geq 1\}.}$$