Vector Calculus Quiz 0409 Solutions
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Problem 1. Find the gradient field of the function
$$f(x,y,z) = \frac{xz+xy+yz}{xyz}.$$

Solution. (A)

We have
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\Big(\frac{z+y}{yz} + \frac{1}{x}\Big) = -\frac{1}{x^2}$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{x+z}{xz} + \frac{1}{y}\Big) = -\frac{1}{y^2}$$
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\Big(\frac{x+y}{xy} + \frac{1}{z}\Big) = -\frac{1}{z^2}$$
Thus $\nabla f = \Big\langle -\frac{1}{x^2}, -\frac{1}{y^2}, -\frac{1}{z^2}\Big\rangle$. □

Problem 2. Let $\vec{F} = \langle x, y\rangle$ and $C : (x+5)^2 + (y-9)^2 = 81$. Calculate the flux of the field $\vec{F}$ across the closed plane curve C,

Solution. (D)

The curve is parameterized by $\vec{r}(t) = \langle -5 + 9\cos t, 9 + 9\sin t\rangle$ with $t \in [0, 2\pi]$, so that $\vec{v}(t) = \langle -9\sin t, 9\cos t\rangle$., and $|\vec{v}| = 9$. The unit normal vector is $\vec{n}(t) = \langle \cos t, \sin t$. Thus
$$\begin{aligned}
\text{Flux} &= \int_C \vec{F}\cdot\vec{n}\,ds \\
&= \int_C \vec{F}\cdot\vec{n}\frac{ds}{dt}\,ds \\
&= \int_C \vec{F}\cdot\vec{n}|\vec{v}|\,dt \\
&= 9\int_0^{2\pi} \langle -5 + 9\cos t, 9 + 9\sin t\rangle \cdot \langle \cos t, \sin t\rangle\,dt \\
&= 9\int_0^{2\pi} -5\cos t + 9\cos^2 t + 9\sin t + 9\sin^2 t\,dt \\
&= 9\int_0^{2\pi} -5\cos t + 9\sin t + 9\,dt \\
&= 9\Big(-5\sin t - 9\cos t + 9t\Big|_0^{2\pi}\Big) \\
&= 162\pi
\end{aligned}$$
□

Problem 3. Let $\vec{F} = \langle y - z, x + 2y - z, -x - y \rangle$. Find the potential function f for the field $\vec{F}$.

Solution. (C)

This means, find f such that $\nabla f = \vec{F} = \langle F_1, F_2, F_3 \rangle$.

First let's integrate F_1 with respect to x to get

$$f(x, y, z) = \int y - z \, dx = xy - xz + g(y, z),$$

where $g(y, z)$ is constant with respect to x, but is a function of y and z.

Next, we look for $g(y, z)$. We know

$$\frac{\partial f}{\partial y} = F_2 \Rightarrow x + \frac{\partial g}{\partial y} = x + 2y - z \Rightarrow \frac{\partial g}{\partial y} = 2y - z.$$

Integrate $\frac{\partial g}{\partial y}$ with respect to y to find that

$$g(y, z) = \int 2y - z \, dy = y^2 - yz + h(z), \text{ so } f(x, y, z) = xy - xz + y^2 - yz + h(z),$$

where $h(z)$ is a function only of z.

Finally, we know

$$\frac{\partial f}{\partial z} = F_3 \Rightarrow -x - y + \frac{dh}{dz} = -x - y, \text{ so } \frac{dh}{dz} = 0, \text{ so } h(z) = C$$

for some constant C.

Putting this together gives

$$f(x, y, z) = xy - xz + y^2 - yz + C.$$

□

Problem 4. Given that the differential is exact, compute

$$\int_{(0,0,0)}^{(4,6,2)} (2xy^2 - 2xz^2) \, dx + 2x^2 y \, dy - 2x^2 z \, dz.$$

Solution. Since the differential is exact, the vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle = \langle 2xy^2 - 2xz^2, 2x^2 y, -2x^2 z \rangle$ is conservative, so there exists a potential function f such that $\nabla f = \vec{F}$. If we find f, we can use the Theorem 1 in Section 16.3 to compute the integral.

In a manner similar to Problem 3, we find a potential for $\vec{F}$; the reader can check that

$$f(x, y, z) = x^2 y^2 - x^2 z^2 + C.$$

Then

$$\int_{(0,0,0)}^{(4,6,2)} (2xy^2 - 2xz^2) \, dx + 2x^2 y \, dy - 2x^2 z \, dz = f(4, 6, 2) - f(0, 0, 0) = 512.$$

□

Problem 5. Find the outward flux of $\vec{F} = \langle -\sqrt{x^2+y^2}, \sqrt{x^2+y^2} \rangle$ across the closed curve C, given as the boundary of the region described in polar coordinates as $1 \le r \le 4$ and $0 \le \theta \le \pi$.

Solution. (D)

Green's Theorem says the the flux across the boundary equals the integral of divergence on the interior. So we compute the divergence to be

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = -\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}.$$

We use polar coordinates to compute

$$\begin{aligned}
\text{Flux} &= \int_C \vec{F} \cdot \vec{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA \\
&= \iint_R -\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\, dA \\
&= \int_0^\pi \int_1^4 -\frac{r\cos\theta}{r} + \frac{r\sin\theta}{r} r\, dr\, d\theta \\
&= \left(\int_1^4 r\, dr \right) \cdot \left(\int_0^\pi \sin\theta - \cos\theta\, d\theta \right) \\
&= \Big[-\cos\theta - \sin\theta \Big]_0^\pi \cdot \left[\frac{r^2}{2} \right]_1^4 \\
&= \Big[(1-0) - (-1-0) \Big] \cdot \left[\frac{16}{2} - \frac{1}{2} \right] \\
&= 15.
\end{aligned}$$

□