AP Calculus AB
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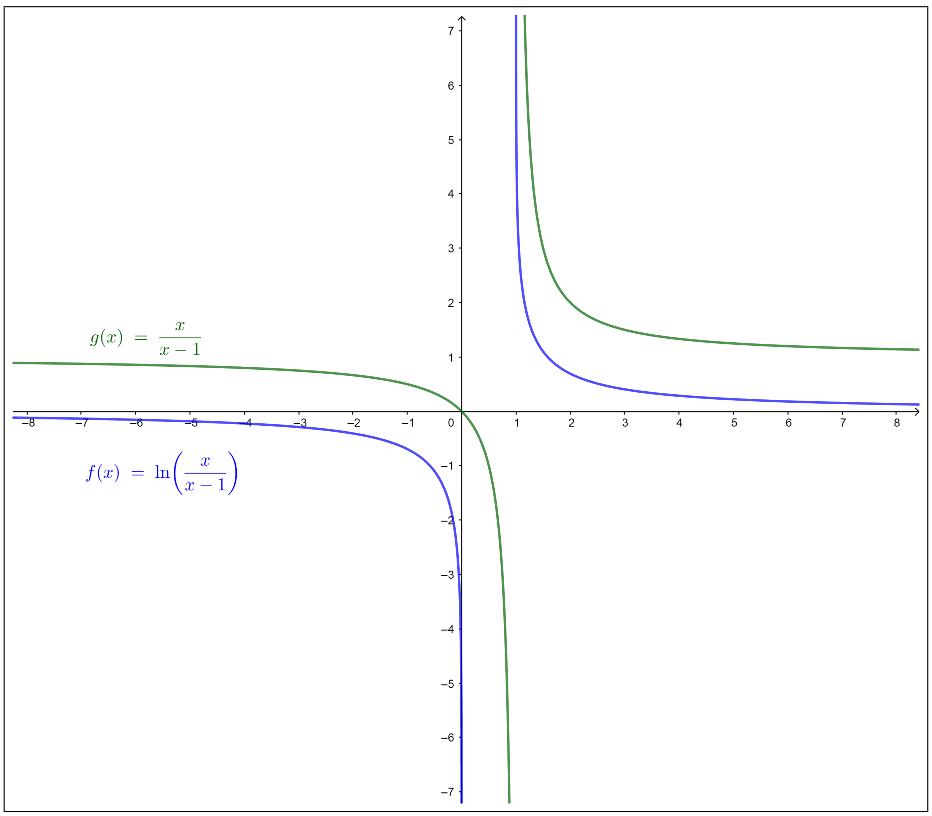
Homework 0413 Solutions
Monday, April 13, 2020

If you would like full credit on the AP Calculus AB solutions during the actual exam, *please* justify all your answers, and write with words and sentences whenever it makes sense to do so.

Problem 1. Let f be the function given by $f(x) = \ln\left(\dfrac{x}{x-1}\right)$.

(a) What is the domain of f?

Solution. Consider the function $g(x) = \dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$; it's graph is obtained by shifting $\dfrac{1}{x}$ right by 1 and up by 1. So, there is a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 1$. Now $f(x) = \ln(g(x))$, so $f(x)$ is positive if $g(x) > 1$, $f(x)$ is zero if $g(x) = 1$, $f(x)$ is negative if $0 < g(x) < 1$, and $f(x)$ is undefined if $g(x) \leq 0$. Use this information to sketch the graphs of $f(x)$ and $g(x)$.



From the graph, we see that:

- f and g are both undefined at $x = 1$;
- $g(x) = 0$ at $x = 0$, so f is undefined there;
- $g(x) < 0$ if $x \in (0, 1)$, so f is undefined there;
- $g(x) > 0$ if $x \in (-\infty, 0) \cup (1, \infty)$.

Therefore,

$$\boxed{\text{dom}(f) = (-\infty, 0) \cup (1, \infty)}.$$

□

(b) Find the value of the derivative of f at $x = -1$.

Solution. We could try to be clever and write $g(x) = \ln(x) - \ln(x-1)$, so $g'(x) = \frac{1}{x} - \frac{1}{x-1}$, so $g'(-1) = -1 - \frac{1}{-1-1} = -\frac{1}{2}$. But since -1 is not in the domain of the expression we just took the derivative of, can we be certain this is right? Better to double check.

We use the chain rule and the product rule to compute

$$f'(x) = \frac{1}{x/(x-1)} \cdot \frac{1 \cdot (x-1) - x(1)}{(x-1)^2} = -\frac{1}{x(x-1)},$$

so

$$f'(-1) = -\frac{1}{(-1)(-2)} = \boxed{-\frac{1}{2}.}$$

□

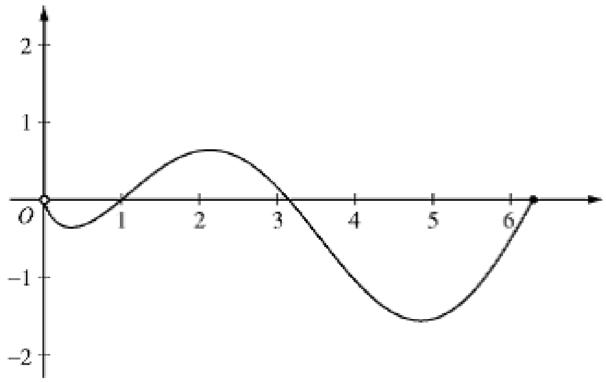
(c) Write an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of f. We let $y = f(x) = \ln\left(\frac{x}{x-1}\right)$, then switch the x and y, and solve for y.

$$\begin{aligned}
y &= \ln\left(\frac{x}{x-1}\right)\\
x &= \ln\left(\frac{y}{y-1}\right)\\
e^x &= \frac{y}{y-1} = \frac{y-1+1}{y-1} = 1 + \frac{1}{y-1}\\
e^x - 1 &= \frac{1}{y-1}\\
y - 1 &= \frac{1}{e^x - 1}\\
y &= 1 + \frac{1}{e^x - 1}
\end{aligned}$$

Thus

$$\boxed{f^{-1}(x) = 1 + \frac{1}{e^x - 1}.}$$

Problem 2. Let f be the function given by $f(x) = (\ln x)(\sin x)$, shown for $0 < x \leq 2\pi$.



Graph of f

The function g is defined by $g(x) = \displaystyle\int_1^x f(t)\,dt$ for $0 < x < 2\pi$.

(a) Find $g(1)$ and $g'(1)$.

Solution. We see that $g(1) = \int_1^1 f(t)\,dt = 0$, from the definition, and that $g'(1) = 0$ from the graph. □

(b) On what intervals, if any, is g increasing? Justify your answer.

Solution. We know g is increasing whenever $g' = f$ is positive, which from the graph looks to be for x between 1 and some number a little bigger than 3. What on earth could that number be?

Well it looks like π and it smells like π (apple) ... but can we verify that it is π?

Yes, indeed, we have $g'(\pi) = \ln(\pi)\sin(\pi) = \ln(\pi) \cdot 0 = 0$.

So g is increasing on $(1, \pi)$. □

(c) For $0 < x \leq 2\pi$, find the value of x at which g has an absolute minimum. Justify your answer.

Solution. The absolute minimum occurs at 2π. We know that g has a local minimum at $x = 1$, because the derivative changes from negative to positive there.

Since the graph of f is above the x-axis on $(1, \pi)$ and below the x-axis on $(\pi, 2\pi)$, we see that $g(2\pi) = g(1) + \int_1^\pi |f(x)|\,dx - \int_\pi^{2\pi} |f(x)|\,dx$. However, we are subtracting more than we are adding here, because the area between the graph and the axis is larger between π and 2π than it is between 1 and π. Thus the absolute minimum occurs at the endpoint. □

(d) For $0 < x \leq 2\pi$, is there a value of x at which the graph of g is tangent to the x-axis? Explain why or why not.

Solution. Yes, since $g(1) = 0$ and $g'(1) = 0$, we know that g touches the x axis where it has a local minimum. □