Vector Calculus — Homework 0413 Solutions

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We are confronted with this formula in the book:

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

What the heck does that mean again?

Firstly, one should note that the circle on the integral sign says nothing more nor less than that C is a closed curve. If you already know that, or if it doesn't effect the computation in any way, it is pointless.

Let $\vec{F} = \langle F_1, F_2 \rangle$, where F_1 and F_2 are scalar valued function defined on a simply connected subset of $\mathbb{R}^2$. Let $\vec{r}$ parameterize the curve C, and let $\vec{v} = \frac{d\vec{r}}{dt}$. Then

$$\begin{aligned}
\int F_1\,dx + F_2\,dy &= \int F_1\frac{dx}{dt} + F_2\frac{dy}{dt}\,dt \\
&= \int \Big\langle F_1, F_2 \Big\rangle \cdot \Big\langle \frac{dx}{dt}, \frac{dy}{dt} \Big\rangle\,dt \\
&= \int \vec{F}\cdot\vec{v}\,dt \\
&= \int \vec{F}\cdot\vec{T}|\vec{v}|\,dt && \text{where } \vec{T} \text{ is the unit tangent vector} \\
&= \int \vec{F}\cdot\vec{T}\,ds \\
&= \text{Flow of } \vec{F} \text{ along } C
\end{aligned}$$

So this weird looking thing is just a flow integral, and that is how we compute it.

Okay so what about $F_1\,dy - F_2\,dx$? In this case, write

$$\begin{aligned}
\int F_1\,dy - F_2\,dx &= \int F_1\frac{dy}{dt} - F_2\frac{dx}{dt}\,dt \\
&= \int \Big\langle F_1, F_2 \Big\rangle \cdot \Big\langle \frac{dy}{dt}, -\frac{dx}{dt} \Big\rangle\,dt \\
&= \int \vec{F}\cdot\vec{w}\,dt && \text{where } \vec{w} \text{ is a normal vector of length } |\vec{v}| \\
&= \int \vec{F}\cdot\vec{n}|\vec{v}|\,dt && \text{where } \vec{n} \text{ is a unit normal vector} \\
&= \int \vec{F}\cdot\vec{n}\,ds \\
&= \text{Flux of } \vec{F} \text{ across } C
\end{aligned}$$

Problem 1 (Thomas §16.4 # 23)**.** Find the area enclosed by the astroid $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$.

Solution. Let C be the image of $\vec{r}$ and R be the region enclosed by C. Let $\vec{F} = \langle x, y \rangle$.

We see that $\vec{v}(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle$, so the normal vector of the same length is $\vec{w} = \langle 3\sin^2 t \cos t, 3\cos^2 t \sin t \rangle$. Then

$$\begin{aligned}
\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} \vec{F}\cdot\vec{w}\,dt \\
&= \frac{1}{2}\int_0^{2\pi} \langle \cos^3 t, \sin^3 t \rangle \cdot \langle 3\sin^2 t \cos t, 3\cos^2 t \sin t \rangle\,dt \\
&= \frac{1}{2}\int_0^{2\pi} 3\sin^2 t\cos^4 t + 3\sin^4 t\cos^2 t\,dt \\
&= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)\,dt \\
&= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt \\
&= \frac{3}{4}\int_0^{2\pi} \sin^2 2t\,dt \\
&= \frac{3}{4}\left[\frac{t}{2} - \frac{\sin 4t}{4}\right]_0^{2\pi} \\
&= \frac{3}{8}\pi.
\end{aligned}$$

Here is an alternate way to write this. Since $x = \cos^3 t$, we have $dx = -3\cos^2 t \sin t\,dt$. Since $y = \sin^3 t$, we have $dy = 3\sin^2 t\cos t\,dt$. Plug in dx and dy to get

$$\begin{aligned}
\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} \cos^3 t(3\sin^2 t\cos t) - \sin^3 t(-3\cos^2 t\sin t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} 3\sin^2 t\cos^4 t + 3\sin^4 t\cos^2 t\,dt \\
&= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)\,dt \\
&= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt \\
&= \frac{3}{4}\int_0^{2\pi} \sin^2 2t\,dt \\
&= \frac{3}{4}\left[\frac{t}{2} - \frac{\sin 4t}{4}\right]_0^{2\pi} \\
&= \frac{3}{8}\pi.
\end{aligned}$$

□

Problem 2. Let $f(x,y) = \ln(x^2 + y^2)$.

(a) Let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \vec{n}\, ds.$$

(b) Let K be a simple closed curve which does not pass through $(0,0)$. Show that

$$\oint_K \nabla f \cdot \vec{n}\, ds$$

can have two possible values, depending on whether $(0,0)$ lies inside K or outside K.

Solution. The gradient is

$$\nabla f = \vec{F} = \langle F_1, F_2 \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle.$$

The curve C does not reside in a simply connected region in the domain of f, and therefore, Green's Theorem does not apply. We compute directly.

The circle is parameterized by $\vec{r}(t) = \langle a\cos t, a\sin t\rangle$. The normal vector of the same length is the position vector, so let $\vec{w} = \vec{r}$. Along the curve, we have

$$\nabla f = \left\langle \frac{2\cos t}{a}, \frac{2\sin t}{a} \right\rangle,$$

so

$$\begin{aligned}
\oint_C \nabla f \cdot \vec{n}\, ds &= \int_C \nabla f \cdot \vec{w}\, dt \\
&= \int_0^{2\pi} \left\langle \frac{2\cos t}{a}, \frac{2\sin t}{a} \right\rangle \cdot \langle a\cos t, a\sin t\rangle\, dt \\
&= \int_0^{2\pi} 2\cos^2 t + 2\sin^2 t\, dt \\
&= \int_0^{2\pi} 2\, dt \\
&= 4\pi.
\end{aligned}$$

Note that this is independent of a.

Now ∇f is conservative on any simply connected domain in which it is defined. If the region bounded by K contains the origin, it can be continuously deformed into a circle through a region in which ∇f is defined, and so,

$$\oint_K \nabla f \cdot \vec{n}\, ds = \oint_C \nabla f \cdot \vec{n}\, ds = 4\pi.$$

On the other hand, if K does not encircle the origin, it can be continuously deformed to a point through a region in which ∇f is defined, in which case we have

$$\oint_K \nabla f \cdot \vec{n}\, ds = 0.$$

It is worth noting that if we eliminate the requirement that K be a *simple* closed curve, and allow it to wrap around the origin m times, then

$$\oint_K \nabla f \cdot \vec{n}\, ds = 4\pi m.$$

In this way, we can detect the wrapping number for a curve. □