Mathematics Learning Centre



Integration: Using the chain rule in reverse

Mary Barnes

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1 Using the Chain Rule in Reverse

Recall that the Chain Rule is used to differentiate composite functions such as $\cos(x^3+1), e^{\frac{1}{2}x^2}, (2x^2+3)^{11}, \ln(3x+1)$. (The Chain Rule is sometimes called the Composite Functions Rule or Function of a Function Rule.)

If we observe carefully the answers we obtain when we use the chain rule, we can learn to recognise when a function has this form, and so discover how to integrate such functions.

Remember that, if y = f(u) and u = g(x)

 $\frac{dy}{dx}$

so that y = f(g(x)), (a composite function)

then

$$=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Using function notation, this can be written as

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In this expression, f'(g(x)) is another way of writing $\frac{dy}{du}$ where y = f(u) and u = g(x)and g'(x) is another way of writing $\frac{du}{dx}$ where u = g(x).

This last form is the one you should learn to recognise.

Examples

By differentiating the following functions, write down the corresponding statement for integration.

i. $\sin 3x$

ii. $(2x+1)^7$

iii. e^{x^2}

Solution

i
$$\frac{d}{dx}\sin 3x = \cos 3x \cdot 3$$
, so $\int \cos 3x \cdot 3dx = \sin 3x + c$.
ii $\frac{d}{dx}(2x+1)^7 = 7(2x+1)^6 \cdot 2$, so $\int 7(2x+1)^6 \cdot 2dx = (2x+1)^7 + c$.
iii $\frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot 2x$, so $\int e^{x^2} \cdot 2xdx = e^{x^2} + c$.

Exercises 1.1

Differentiate each of the following functions, and then rewrite each result in the form of a statement about integration.

i
$$(2x-4)^{13}$$
 ii $\sin \pi x$ iii e^{3x-5}
iv $\ln(2x-1)$ v $\frac{1}{5x-3}$ vi $\tan 5x$
vii $(x^5-1)^4$ viii $\sin(x^3)$ ix $e^{\sqrt{x}}$
x $\cos^5 x$ xi $\tan(x^2+1)$ xii $\ln(\sin x)$

The next step is to learn to *recognise* when a function has the forms $f'(g(x)) \cdot g'(x)$, that is, when it is the derivative of a composite function. Look back at each of the integration statements above. In every case, the function being integrated is the *product* of two functions: one is a composite function, and the other is the derivative of the "inner function" in the composite. You can think of it as "the derivative of what's inside the brackets". Note that in some cases, this derivative is a constant.

For example, consider

$$\int e^{3x} \cdot 3dx.$$

We can write e^{3x} as a composite function. 3 is the derivative of 3x i.e. the derivative of "what's inside the brackets" in $e^{(3x)}$.

This is in the form

$$\int f'(g(x)) \cdot g'(x) dx$$

with

$$u = g(x) = 3x$$
, and $f'(u) = e^u$

Using the chain rule in reverse, since $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ we have

$$\int f'(g(x)) \cdot g'(x) dx = f(g(x)) + c.$$

In this case

$$\int e^{3x} \cdot 3dx = e^{3x} + c.$$

If you have any doubts about this, it is easy to check if you are right: *differentiate your* answer!

Now let's try another:

$$\int \cos(x^2 + 5) \cdot 2x dx.$$

 $\cos(x^2+5)$ is a composite function.

2x is the derivative of $x^2 + 5$, i.e. the derivative of "what's inside the brackets".

So this is in the form

$$\int f'(g(x)) \cdot g'(x) dx \quad \text{with } u = g(x) = x^2 + 5 \text{ and } f'(u) = \cos u$$

Recall that if $f'(u) = \cos u$, $f(u) = \sin u$.

So,

$$\int \cos(x^2 + 5) \cdot 2x dx = \sin(x^2 + 5) + c$$

Again, check that this is correct, by differentiating.

People sometimes ask "Where did the 2x go?". The answer is, "Back where it came from."

If we differentiate $\sin(x^2 + 5)$ we get $\cos(x^2 + 5) \cdot 2x$.

So when we integrate $\cos(x^2 + 5) \cdot 2x$ we get $\sin(x^2 + 5)$.

Examples

Each of the following functions is in the form $f'(g(x)) \cdot g'(x)$.

Identify f'(u) and u = g(x) and hence find an indefinite integral of the function.

i. $(3x^2 - 1)^4 \cdot 6x$

ii.
$$\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

Solutions

i. $(3x^2 - 1)^4 \cdot 6x$ is a product of $(3x^2 - 1)^4$ and 6x.

Clearly $(3x^2 - 1)^4$ is the composite function f'(g(x)). So g(x) should be $3x^2 - 1$. 6x is the "other part". This should be the derivative of "what's inside the brackets" i.e. $3x^2 - 1$, and clearly, this is the case:

$$\frac{d}{dx}(3x^2 - 1) = 6x.$$

So, $u = g(x) = 3x^2 - 1$ and $f'(u) = u^4$ giving $f'(g(x)) \cdot g'(x) = (3x^2 - 1)^4 \cdot 6x.$
If $f'(u) = u^4$, $f(u) = \frac{1}{5}u^5.$
So, using the rule

$$\int f'(g(x)) \cdot g'(x) dx = f(g(x)) + c$$

we conclude

$$\int (3x^3 - 1)^4 \cdot 6x = \frac{1}{5}(3x^2 - 1)^5 + c$$

You should differentiate this answer *immediately* and check that you get back the function you began with.

ii. $\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

This is a product of $\sin(\sqrt{x})$ and $\frac{1}{2\sqrt{x}}$.

Clearly $\sin(\sqrt{x})$ is a composite function.

The part "inside the brackets" is \sqrt{x} , so we would like this to be g(x). The other factor $\frac{1}{2\sqrt{x}}$ ought to be g'(x). Let's check if this is the case:

$$g(x) = \sqrt{x} = x^{\frac{1}{2}}$$
, so $g'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

So we're right! Thus $u = g(x) = \sqrt{x}$ and $f'(u) = \sin u$ giving

$$f'(g(x)) \cdot g'(x) = \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

Now, if $f'(u) = \sin u$, $f(u) = -\cos u$.

So using the rule

$$\int f'(g(x)) \cdot g'(x) dx = f(g(x)) + c$$

we conclude

$$\int \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} dx = -\cos(\sqrt{x}) + c.$$

Again, check immediately by differentiating the answer.

Note: The explanations given here are fairly lengthy, to help you to understand what we're doing. Once you have grasped the idea, you will be able to do these very quickly, without needing to write down any explanation.

Example

Integrate $\int \sin^3 x \cdot \cos x dx$.

Solution

$$\int \sin^3 x \cdot \cos x dx = \int (\sin x)^3 \cdot \cos x dx.$$

So $u = g(x) = \sin x$ with $g'(x) = \cos x.$
And $f'(u) = u^3$ giving $f(u) = \frac{1}{4}u^4.$
Hence $\int \sin^3 x \cdot \cos x dx = \frac{1}{4}(\sin x)^4 + c = \frac{1}{4}\sin^4 x + c.$

Exercises 1.2

Each of the following functions is in the form $f'(g(x)) \cdot g'(x)$. Identify f'(u) and u = g(x) and hence find an indefinite integral of the function.

i	$\frac{1}{3x-1} \cdot 3$	ii	$\sqrt{2x+1} \cdot 2$	iii	$(\ln x)^2 \cdot \frac{1}{x}$
iv	$e^{2x+4} \cdot 2$	v	$\sin(x^3) \cdot 3x^2$	vi	$\cos\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2}$
vii	$(7x-8)^{12} \cdot 7$	viii	$\sin(\ln x) \cdot \frac{1}{x}$	ix	$\left(\frac{1}{\sin x}\right) \cdot \cos x$
x	$e^{\tan x} \cdot \sec^2 x$	xi	$e^{x^3} \cdot 3x^2$	xii	$\sec^2(5x-3)\cdot 5$
xiii	$(2x-1)^{\frac{1}{3}} \cdot 2$	xiv	$\sqrt{\sin x} \cdot \cos x$		

The final step in learning to use this process is to be able to recognise when a function is not quite in the correct form but *can be put into the correct form* by minor changes.

For example, we try to calculate $\int x^3 \sqrt{x^4 + 1} dx$.

We notice that $\sqrt{x^4 + 1}$ is a composite function, so we would like to have $u = g(x) = x^4 + 1$. But this would mean $g'(x) = 4x^3$, and the integrand (i.e. the function we are trying to integrate) only has x^3 . However, we can easily make it $4x^3$, as follows:

$$\int x^3 \sqrt{x^4 + 1} dx = \frac{1}{4} \int \sqrt{x^4 + 1} \cdot 4x^3 dx.$$

Note: The $\frac{1}{4}$ and the 4 cancel with each other, so the expression is not changed.

So
$$u = g(x) = x^4 + 1$$
, $g'(x) = 4x^3$
And $f'(u) = u^{\frac{1}{2}}$ $f(u) = \frac{2}{3}u^{\frac{3}{2}}$
So, $\int x^3 \sqrt{x^4 + 1} dx = \frac{1}{4} \int \sqrt{x^4 + 1} \cdot 4x^2 dx = \frac{1}{4} \cdot \frac{2}{3} \left(x^4 + 1\right)^{\frac{3}{2}} + c$.

Note: We may only insert constants in this way, not variables.

We cannot for example evaluate $\int e^{x^2} dx$ by writing $\frac{1}{2x} \int e^{x^2} \cdot 2x dx$, because the $\frac{1}{x}$ in front of the integral sign does not cancel with the x which has been inserted in the integrand.

This integral cannot, in fact, be evaluated in terms of elementary functions.

The example above illustrates one of the difficulties with integration: many seemingly simple functions cannot be integrated without inventing new functions to express the integrals. There is no set of rules which we can apply which will tell us how to integrate any function. All we can do is give some techniques which will work for some functions.

Exercises 1.3

Write the following functions in the form $f'(g(x)) \cdot g'(x)$ and hence integrate them:

i	$\cos 7x$	ii	xe^{x^2}
iii	$\frac{x}{1-2x^2}$	iv	$x^2(4x^3+3)^9$
\mathbf{v}	$\sin(1+3x)$	vi	$\frac{\sin\sqrt{x}}{\sqrt{x}}$
vii	$\frac{x}{\sqrt{(1-x^2)}}$	viii	e^{3x}
ix	$\tan 6x$		

Hint: Write $\tan 6x$ in terms of $\sin 6x$ and $\cos 6x$.

2 Solutions to exercises

Exercises 1.1

Exercises 1.2

(Before you read these solutions, *check* your work by differentiating your answer.)

i.
$$\int \frac{1}{3x - 1} \cdot 3dx = \ln(3x - 1) + c.$$

$$\begin{cases} u = g(x) = 3x - 1 & \text{so } g'(x) = 3 \\ f'(u) = \frac{1}{u} & \text{so } f(u) = \ln u \end{cases}$$
ii.
$$\int \sqrt{2x + 1} \cdot 2dx = \frac{2}{3}(2x + 1)^{\frac{3}{2}} + c.$$

$$\begin{cases} u = g(x) = 2x + 1 & \text{so } g'(x) = 2 \\ f'(u) = \sqrt{u} & \text{so } f(u) = \frac{2}{3}u^{\frac{3}{2}} \end{cases}$$
iii.
$$\int (\ln x)^2 \cdot \frac{1}{x} dx = \frac{1}{3}(\ln x)^3 + c.$$

$$\begin{cases} u = g(x) = \ln x & \text{so } g'(x) = \frac{1}{x} \\ f'(u) = u^2 & \text{so } f(u) = \frac{1}{3}u^3 \end{cases}$$
iv.
$$\int e^{2x+4} \cdot 2dx = e^{2x+4} + c.$$

$$\begin{cases} u = g(x) = 2x + 4 & \text{so } g'(x) = 2 \\ f'(u) = e^u & \text{so } f(u) = e^u \end{cases}$$
v.
$$\int \sin(x^3) \cdot 3x^2 dx = -\cos(x^3) + c.$$

$$\begin{cases} u = g(x) = x^3 & \text{so } g'(x) = 3x^2 \\ f'(u) = \sin u & \text{so } f(u) = -\cos u \end{cases}$$
vi.
$$\int \cos(\frac{\pi x}{2}) \cdot \frac{\pi}{2} dx = \sin(\frac{\pi x}{2}) + c.$$

$$\begin{cases} u = g(x) = \frac{\pi}{2}x & \text{so } g'(x) = \frac{\pi}{2} \\ f'(u) = \cos u & \text{so } f(u) = -\cos u \end{cases}$$
vii.
$$\int \cos(\frac{\pi x}{2}) \cdot \frac{\pi}{2} dx = \sin(\frac{\pi x}{2}) + c.$$

$$\begin{cases} u = g(x) = \frac{\pi}{2}x & \text{so } g'(x) = \frac{\pi}{2} \\ f'(u) = \cos u & \text{so } f(u) = \sin u \end{cases}$$
vii.
$$\int (7x - 8)^{12} \cdot 7 dx = \frac{1}{13}(7x - 8)^{13} + c.$$

$$\begin{cases} u = g(x) = 7x - 8 & \text{so } g'(x) = 7 \\ f'(u) = u^{12} & \text{so } f(u) = \frac{1}{13}u^{13} \end{cases}$$
viii.
$$\int \sin(\ln x) \cdot \frac{1}{x} dx = -\cos(\ln x) + c.$$

$$\begin{cases} u = g(x) = \ln x & \text{so } g'(x) = \frac{1}{x} \\ f'(u) = \sin u & \text{so } f(u) = -\cos u \end{cases}$$
ix.
$$\int \frac{1}{\sin x} \cdot \cos x dx = \ln(\sin x) + c.$$

$$\begin{cases} u = g(x) = \sin x & \text{so } g'(x) = \cos x \\ f'(u) = \frac{1}{u} & \text{so } f(u) = \ln u \end{cases}$$
x.
$$\int e^{\tan x} \cdot \sec^2 x dx = e^{\tan x} + c.$$

$$\begin{cases} u = g(x) = \tan x & \text{so } g'(x) = \sec^2 x \\ f'(u) = e^u & \text{so } f(u) = e^u \end{cases}$$

$$\begin{aligned} \mathbf{xi.} & \int e^{x^3} \cdot 3x^2 dx = e^{x^3} + c. \\ & \begin{cases} u = g(x) = x^3 & \text{so } g'(x) = 3x^2 \\ f'(u) = e^u & \text{so } f(u) = e^u \end{cases} \\ & \mathbf{xii.} & \int \sec^2(5x - 3) \cdot 5dx = \tan(5x - 3) + c. \\ & \begin{cases} u = g(x) = 5x - 3 & \text{so } g'(x) = 5 \\ f'(u) = \sec^2 u & \text{so } f(u) = \tan u \end{cases} \\ & \mathbf{xiii.} & \int (2x - 1)^{\frac{1}{3}} \cdot 2dx = \frac{3}{4}(2x - 1)^{\frac{4}{3}} + c. \\ & \begin{cases} u = g(x) = 2x - 1 & \text{so } g'(x) = 2 \\ f'(u) = u^{\frac{1}{3}} & \text{so } f(u) = \frac{3}{4}u^{\frac{4}{3}} \end{cases} \\ & \mathbf{xiv.} & \int \sqrt{\sin x} \cdot \cos x dx = \frac{2}{3}(\sin x)^{\frac{3}{2}} + c. \\ & \begin{cases} u = g(x) = \sin x & \text{so } g'(x) = \cos x \\ f'(u) = \sqrt{u} & \text{so } f(u) = \frac{2}{3}u^{\frac{3}{2}} \end{aligned} \end{aligned}$$

Exercises 1.3

(Before reading the solutions, check all your answers by differentiating!)

$$\begin{aligned} \mathbf{i.} & \int \cos 7x dx = \frac{1}{7} \int \cos 7x \cdot 7 dx = \frac{1}{7} \sin 7x + c. \\ & \begin{cases} u = g(x) = 7x, \ g'(x) = 7 \\ f'(u) = \cos u \text{ so } f(u) = \sin u \end{aligned}$$
$$\\ \mathbf{ii.} & \int x e^{x^2} dx = \frac{1}{2} \int e^{x^2} \cdot 2x dx = \frac{1}{2} e^{x^2} + c. \\ & \begin{cases} u = g(x) = x^2, \ g'(x) = 2x \\ f'(u) = e^u \text{ so } f(u) = e^u \end{aligned}$$
$$\\ \mathbf{iii.} & \int \frac{x}{1 - 2x^2} dx = -\frac{1}{4} \int \frac{1}{1 - 2x^2} \cdot (-4x) dx = -\frac{1}{4} \ln(1 - 2x^2) + c. \\ & \begin{cases} u = g(x) = 1 - 2x^2, \ g'(x) = -4x \\ f'(u) = \frac{1}{u} \text{ so } f(u) = \ln u \end{aligned}$$
$$\\ \mathbf{iv.} & \int x^2 (4x^3 + 3)^9 dx = \frac{1}{12} \int (4x^3 + 3)^9 \cdot 12x^2 dx = \frac{1}{12} \cdot \frac{1}{10} (4x^3 + 3)^{10} + c = \frac{1}{120} (4x^3 + 3)^{10} + c. \\ & \begin{cases} u = g(x) = 4x^3 + 3, \ g'(x) = 12x^2 \\ f'(u) = u^9 \text{ so } f(u) = \frac{1}{10} u^{10} \end{aligned}$$

$$\mathbf{v}. \quad \int \sin(1+3x)dx = \frac{1}{3} \int \sin(1+3x) \cdot 3dx = -\frac{1}{3}\cos(1+3x) + c. \\ \begin{cases} u = g(x) = 1 + 3x, \ g'(x) = 3 \\ f'(u) = \sin u \ \mathbf{so} \ f(u) = -\cos u \end{cases}$$

$$\mathbf{vi.} \quad \int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2 \int \sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}dx = -2\cos\sqrt{x} + c. \\ \begin{cases} u = g(x) = \sqrt{x}, \ g'(x) = \frac{1}{2\sqrt{x}} \\ f'(u) = \sin u \ \mathbf{so} \ f(u) = -\cos u \end{cases}$$

$$\mathbf{vii.} \quad \int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \cdot (-2x)dx = -\frac{1}{2} \cdot 2(1-x^2)^{\frac{1}{2}} + c = -(1-x^2)^{\frac{1}{2}} + c. \\ \begin{cases} u = g(x) = 1 - x^2, \ g'(x) = -2x \\ f'(u) = \frac{1}{\sqrt{u}} \ \mathbf{so} \ f(u) = 2u^{\frac{1}{2}} \end{cases}$$

$$\mathbf{viii.} \quad \int e^{3x}dx = \frac{1}{3} \int e^{3x} \cdot 3dx = \frac{1}{3}e^{3x} + c. \\ \begin{cases} u = g(x) = 3x, \ g'(x) = 3 \\ f'(u) = e^{u} \ \mathbf{so} \ f(u) = e^{u} \end{cases}$$

$$\mathbf{ix.} \quad \int \tan 6xdx = \int \frac{\sin 6x}{\cos 6x}dx = -\frac{1}{6} \int \frac{1}{\cos 6x} \cdot -6\sin 6x = -\frac{1}{6}\ln(\cos 6x) + c. \\ \end{cases} \\ \begin{cases} u = g(x) = \cos 6x, \ g'(x) = -6\sin 6x \\ f'(u) = \frac{1}{u} \ \mathbf{so} \ f(u) = \ln u \end{cases}$$