AP Calculus AB — Homework 0414 Solutions

Dr. Paul L. Bailey — Tuesday, April 14, 2020

Problem 1. Let f be a function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant

(a) Find $f'(x)$ and $f''(x)$.

Solution. We have

$$f'(x) = \frac{k}{2\sqrt{x}} - \frac{1}{x}.$$

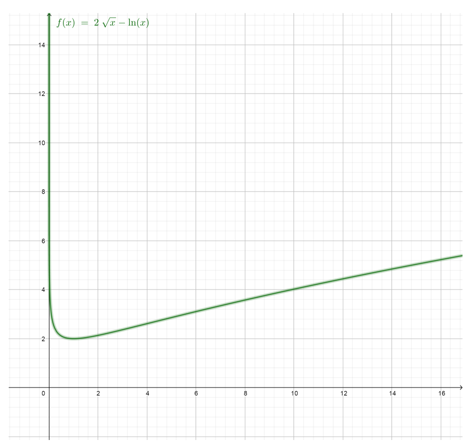
If we rewrite $f'(x) = \frac{k}{2}x^{-1/2} - x^{-1}$, taking the second derivative is easier; we get

$$f''(x) = -\frac{k}{4}x^{-3/2} + x^{-2}.$$

□

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

Solution. If $f'(1) = \frac{k}{2} - 1 = 0$, then $k = 2$. Setting $k = 2$, we wish to classify the critical point that f has at $x = 1$. We use the Second Derivative Test to do this. Now, with $k = 2$, we have $f''(1) = -\frac{2}{4} + 1 > 0$, so f is concave up at $x = 1$, so f has a local minimum at $x = 1$. Here is a graph of f when $k = 2$.



□

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

Solution. We know f has a point of inflection where its second derivative is zero and changes sign. This is occurring on the x-axis, so at the x we seek, $f(x) = 0$ and $f''(x) = 0$. Let's solve the second equation first; $f''(x) = 0$ implies $-\frac{k}{4}x^{-3/2} + x^{-2} = 0$. Multiply through by x^2 to get $\frac{k}{4}\sqrt{x} = 1$, so $k = \frac{4}{\sqrt{x}}$. Plug this k into $f(x) = 0$ to get

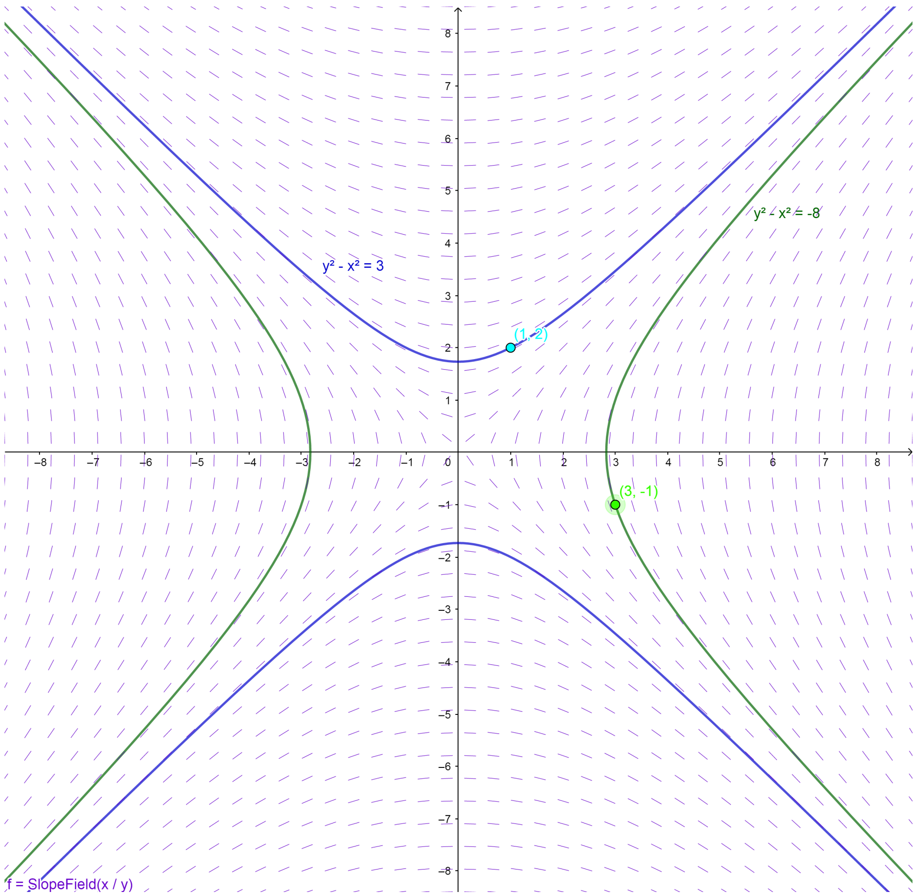
$$\frac{4}{\sqrt{x}}\sqrt{x} - \ln x = 0 \quad \Rightarrow \quad \ln x = 4 \quad \Rightarrow \quad x = e^4,$$

whence $k = \frac{4}{e^2}$.

□

Problem 2. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$, where $y \neq 0$.

(a) Sketch the solution curve that passes through the point $(3, -1)$, and sketch the solution curve that passes through the point $(1, 2)$.



(b) Write an equation for the line tangent to the solution curve that passes through the point $(1, 2)$.

Solution. Note that at $(1, 2)$, we have $\dfrac{dy}{dx} = \dfrac{1}{2}$. The line is $y = \dfrac{1}{2}(x-1) + 2$. ☐

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(3) = -1$, and state its domain.

Solution. This is a separable differential equation with an initial value.

We have $\dfrac{dy}{dx} = \dfrac{x}{y}$, so $y\,dy = x\,dx$, that is, $\int y\,dy = \in x\,dx$. Thus $\dfrac{y^2}{2} = \dfrac{x^2}{2} + C$, and we can multiply through by 2 and let the $k = -2C$ to get $y^2 - x^2 = k$. The solutions are hyperbolas.

Now, let's get the other curve, the one through $(1, 2)$. It's a little easier. If $(1, 2)$ is on the curve, we have $4 - 1 = k$, so the equation of the curve is $y^2 - x^2 = 3$. We can put this in functional form as $f(x) = \sqrt{x^2 + 3}$, and this is defined for all $x \in \mathbb{R}$.

However, let's try it now for the point $(3, -1)$. We get $1 - 9 = k$, so the curve is $y^2 - x^2 = -9$. Solving this for y gives $y = \pm\sqrt{x^2 - 9}$. This is not a function; it has two branches. Since our y-coordinate is negative, we require the lower branch, and set $f(x) = -\sqrt{x^2 - 9}$. The domain is $[3, \infty)$. Notice that this expression is also defined for $x \in (-\infty, -3]$; however, this is not part of the solution, since that piece is disconnected from the piece which contains $(3, -1)$. ☐

Remark 1. Name that thang!

- To return his army to Rome, Caesar crosses the Rubicon.
  The intermediate value theorem states that if f is continuous, $f(a)$ is negative, $f(b)$ is positive, then the graph of f must cross the x axis at some c between a and b. The Rubicon is the x-axis.

- When Pheidippides ran from Marathon to Athens, at some point he ran at his average speed.
  The Mean Value Theorem says the if f is continuous on a closed interval and differentiable on its interior, then the derivative of f equals its average value somewhere in the interval. Pheidippides' instantaneous speed is the derivative of his position, and his average speed is his displacement divided by time.

- Hannibal must have reached his highest point somewhere in the Alps, since it wasn't at the beginning or end of his journey.
  The Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f must have a maximum value on $[a, b]$, and if that maximum does not occur at an endpoint, it must occur at a critical point. Hannibal's critical point was in the Alps.