AP CALCULUS AB — Homework 0415

DR. PAUL L. BAILEY — Tuesday, April 14, 2020

Problem 1. A tank in the shape of a cylinder of radius 5 decimeters and height 20 decimeters, which is filled with water, develops a circular hole in its bottom. The radius of the hole increases at a rate of 2 cm/minute. Water drains from the tank at the rate of 3 liters per minute per square centimeter of area of the hole. How long does it take for the tank to drain completely?

Solution. Keeping track of the unit is important in this problem. A liter is a cubic decimeter. Let t be time in minutes, starting with $t = 0$ when the hole develops. Let r be the radius of the hole and let A be its area. Let V be the volume of water in the tank.

We break the problem down into steps.

(a) Find the initial volume of the tank.
The volume of a circular cylinder of radius 5 and height 20 is

$$V = \pi r^2 h = \pi \cdot 5^2 \cdot 20 = 400\pi \text{dm}^3.$$

(b) Find r as a function of t.
Let time begin when the hole first appears. Since $\dfrac{dr}{dt} = 2\dfrac{\text{cm}}{\text{min}}$, and $r(0) = 0$, we see that $r(t) = 2t\text{cm}$.

(c) Find A as a function of t.
We have

$$A = \pi r^2 = 4\pi t^2 \text{cm}^2.$$

(d) Find $\frac{dV}{dt}$ as a function of t (this should be negative).
Here is where we need to be careful with units. We know that the volume is decreasing at a rate of 3 liters per minute per square centimeter. A liter is a cubic decimeter, which is 1000 cubic centimeters. So the rate of drainage is

$$3000\frac{\frac{\text{cm}^3}{\text{min}}}{\text{cm}^2} \cdot 4\pi t^2 \text{cm}^2 = 12000\pi t^2 \frac{\text{cm}^3}{\text{min}} = 12\pi t^2 \frac{\text{dm}}{\text{min}}.$$

The rate of change of water in the tank is negative, so

$$\frac{dV}{dt} = -4\pi t^2 \frac{\text{dm}}{\text{min}}.$$

(e) Find V as a function of time x by noting that

$$V(x) - V(0) = \int_0^x \frac{dV}{dt}\, dt,$$

where $V(x)$ is the volume of water in the tank at time x and $V(0)$ is the initial volume.
We have

$$V(x) = 400\pi + \int_0^x 12\pi t^2\, dt = 400\pi - 4\pi t^3.$$

(f) Find time x such that $V(x) = 0$.
If $V = 0$, then $4\pi t^3 = 400pi$, so

$$\boxed{t = \sqrt[3]{100}.}$$

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