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We use this formula from Thomas:

$$\text{Surface Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA,$$

where $\vec{p}$ is a unit vector normal to R and $\nabla f \cdot p \neq 0$.

If f is of the form $f(x, y, z) = g(x, y) - z = k$, where k is constant, then the surface may be viewed as the graph of the function $z = g(x, y) + k$, and R is then a region in the xy-plane. In this case, $\vec{p}$ is a unit vector perpendicular to the xy-plane; that is, $\vec{p} = \langle 0, 0, 1\rangle$.

Problem 1 (Thomas §16.5 # 1)**.** Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

Solution. Here, $f(x, y, z) = x^2 + y^2 - z$, and the surface is part of the locus of the equation $f(x, y, z) = 0$. The region R is the disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2\}$, and $\vec{p} = \langle 0, 0, 1\rangle$.

Compute $\nabla f = \langle 2x, 2y, -1\rangle$, $|\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$, and $|\nabla f \cdot \vec{p}| = 1$. Thus, converting to polar coordinates,

$$\begin{aligned}
\text{Surface Area} &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA \\
&= \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}r\, dr\, d\theta \\
&= (2\pi)\frac{1}{8} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}8r\, dr \\
&= \left(\frac{\pi}{4}\right)\left(\frac{2}{3}\right)\Big[(4r^2 + 1)^{3/2}\Big]_0^{\sqrt{2}} \\
&= \frac{\pi}{6}[27 - 1] \\
&= \frac{13\pi}{3}.
\end{aligned}$$

□

Problem 2 (Thomas §16.5 # 2). Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.

Solution. Still, $f(x, y, z) = x^2 + y^2 - z$, and the surface is part of the locus of the equation $f(x, y, z) = 0$. The region R is the annulus $\{(x, y) \in \mathbb{R}^2 \mid 2 \leq x^2 + y^2 \leq 6\}$, and $\vec{p} = \langle 0, 0, 1\rangle$.

Again, $\nabla f = \langle 2x, 2y, -1\rangle$, $|\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$, and $|\nabla f \cdot \vec{p}| = 1$. All that changes from above are the limits of integration in polar coordinates:

$$
\begin{aligned}
\text{Surface Area} &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA \\
&= \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA \\
&= \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\
&= (2\pi) \frac{1}{8} \int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2 + 1}\, 8r\, dr \\
&= \left(\frac{\pi}{4}\right)\left(\frac{2}{3}\right)\left[(4r^2 + 1)^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}} \\
&= \frac{\pi}{6}[125 - 27] \\
&= \frac{49\pi}{3}.
\end{aligned}
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□