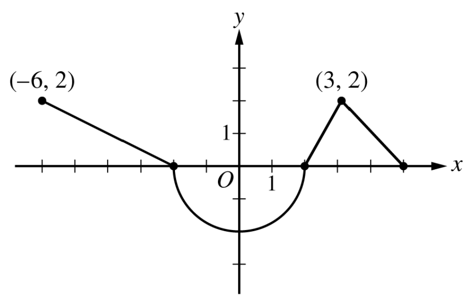
AP Calculus AB — Homework 0420

Dr. Paul L. Bailey — Monday, April 20, 2020

**Problem 1.** The function $f$ is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of $f'$, the derivative of $f$, consists of a semicircle and three line segments, as shown in the figure below.




Graph of $f'$

**(a)** Find the values of $f(-6)$ and $f(5)$.

**(b)** On what intervals is $f$ increasing? Justify your answer.

**(c)** Find the absolute minimum value of $f$ on the closed interval $[-6, 5]$. Justify your answer.

**(d)** For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

*Solution.* We use the Fundamental Theorem of Calculus (FTC): $f(b) = f(a) + \int_a^b f'(x)\,dx$. The integral of $f'$ is the area between the graph of $f'$ and the $x$-axis.

**(a)** Since $f(-2) = 7$,

$$f(-6) = f(-2) - \int_{-6}^{-2} f'(x)\,dx = 7 - \text{area} \underset{[-6,-2]}{\triangle} = 7 - \frac{1}{2}(4)(2) = 7 - 4 = 3,$$

and

$$f(5) = f(-2) + \int_{-2}^{5} = 7 - \text{area} \underset{[-2,2]}{\bigcirc} + \text{area} \underset{[2,5]}{\triangle} = 7 - \frac{1}{2}\pi(2)^2 + \frac{1}{2}(3)(2) = 7 - 2\pi + 3 = 10 - 2\pi.$$

**(b)** The function $f$ is increasing where ever its derivative is positive. This occurs on $[-6, -2]$ and $[2, 5]$.

**(c)** The minimum occurs at a critical point or an endpoint. Since $f$ is differentiable, a critical point occurs when the derivative is zero; that is, at $x = -2$ and $x = 2$. We plug the critical points and endpoints into $f$, and compare values. Now $f(2) = f(-2) - \text{area} \underset{[-2,2]}{\bigcirc} = 7 - 2\pi$ is less than $f(-2) = 7$, $f(-6) = 3$, and $f(5) = 10 - 2\pi$. Thus the minimum value is $7 - 2\pi$.

**(d)** We know that $f''(x)$ is the slope of the tangent line of $f'$ at $x$. Thus, $f''(-5)$ is the slope of the line segment at $x = -5$; that is, $f''(-5) = -\frac{1}{2}$. However, $f'$ is not differentiable at $x = 3$. Indeed, $\lim_{x \to 3^-} f''(x) = 2$ but $\lim_{x \to 3^+} f''(x) = -\frac{3}{2}$. Since the left and right derivatives of $f'$ are not equal at $x = 3$, $f''$ does not exist at $x = 3$.

□