AP Calculus AB — Quiz 0416 Solutions

Dr. Paul L. Bailey — Monday, April 16, 2020

Problem 1. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down.

t (min)	0	2	5	7	11	12
$r'(t)$ (ft/min)	5.7	4.0	2.0	1.2	0.6	0.5

The table above gives selected values of the rate of change, $r'(t)$, of radius of the balloon over the time interval $0 \leq t \leq 12$. The radius of the balloon is 30 feet when $t = 5$.

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

Solution. The point at which the line is tangent is $(t, r) = (5, 30)$. The slope of the line is $r'(5) = 2$. So, the line is

$$\ell(t) = 2(t - 5) + 30.$$

The estimate at $t = 5.4$ is

$$\ell(5.4) = 2(.4) + 30 = 30.8\,\text{ft}.$$

Since r is concave down, $\ell(5.4) \geq r(5.4)$.
(+1 point for value, +1 point for reason) □

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

Solution. The volume of the balloon is $V = \frac{4}{3}\pi r^3$. Taking the derivative with respect to time gives $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Since $r(5) = 30$ and $\frac{dr}{dt}(5) = 2$, we have $\frac{dV}{dt}(2) = 4\pi(900)(2) = 7200\pi\,\frac{\text{ft}^3}{\text{min}}$.
(+2 points for $\frac{dV}{dt}$, +1 point for value, +1 point for units) □

(c) Use a right Riemann sum with the five subintervals indicated by data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

Solution. We have

$$\begin{aligned} R &= \sum_{i=1}^{5} f(t_i)\Delta t_i = 4.0(2-0) + 2.0(5-2) + 1.2(7-5) + 0.6(11-7) + 0.5(12-11) \\ &= 8.0 + 6.0 + 2.4 + 2.4 + 0.5 = 19.3. \end{aligned}$$

The integral is the change in radius, in feet, between 0 minutes and 12 minutes.
(+1 point for value, +1 point for meaning) □

(d) Is your approximation in part **(c)** greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

Solution. Since the graph is concave down, the second derivative is negative, so the derivative is decreasing throughout the interval, whence a right Riemann sum is an underestimate. So, $R < \int_0^{12} r'(t)\,dt$.
(+1 point for conclusion with reason) □