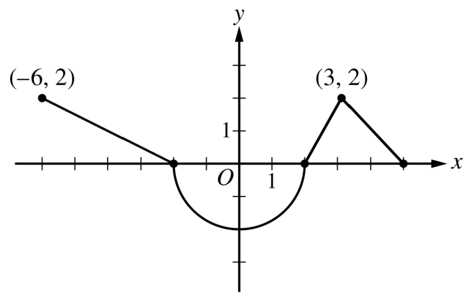
AP Calculus AB
Dr. Paul L. Bailey

Homework 0420
Monday, April 20, 2020

Problem 1. The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure below.



Graph of f'

(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is f increasing? Justify your answer.

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.

(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

Solution. We use the Fundamental Theorem of Calculus (FTC): $f(b) = f(a) + \int_a^b f'(x)\,dx$. The integral of f' is the area between the graph of f' and the x-axis.

(a) Since $f(-2) = 7$,

$$f(-6) = f(-2) - \int_{-6}^{-2} f'(x)\,dx = 7 - \text{area} \underset{[-6,-2]}{\triangle} = 7 - \frac{1}{2}(4)(2) = 7 - 4 = 3,$$

and

$$f(5) = f(-2) + \int_{-2}^{5} = 7 - \text{area} \underset{[-2,2]}{\bigcirc} + \text{area} \underset{[2,5]}{\triangle} = 7 - \frac{1}{2}\pi(2)^2 + \frac{1}{2}(3)(2) = 7 - 2\pi + 3 = 10 - 2\pi.$$

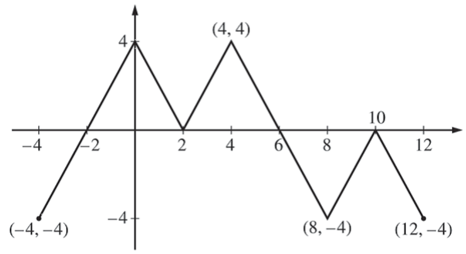
(b) The function f is increasing where ever its derivative is positive. This occurs on $[-6, -2]$ and $[2, 5]$.

(c) The minimum occurs at a critical point or an endpoint. Since f is differentiable, a critical point occurs when the derivative is zero; that is, at $x = -2$ and $x = 2$. We plug the critical points and endpoints into f, and compare values. Now $f(2) = f(-2) - \text{area} \underset{[-2,2]}{\bigcirc} = 7 - 2\pi$ is less than $f(-2) = 7$, $f(-6) = 3$, and $f(5) = 10 - 2\pi$. Thus the minimum value is $7 - 2\pi$.

(d) We know that $f''(x)$ is the slope of the tangent line of f' at x. Thus, $f''(-5)$ is the slope of the line segment at $x = -5$; that is, $f''(-5) = -\dfrac{1}{2}$. However, f' is not differentiable at $x = 3$. Indeed, $\lim_{x \to 3^-} f''(x) = 2$ but $\lim_{x \to 3^+} f''(x) = -\dfrac{3}{2}$. Since the left and right derivatives of f' are not equal at $x = 3$, f'' does not exist at $x = 3$.

□

Problem 2. The figure below shows the graph of the piecewise linear function f. For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \displaystyle\int_2^x f(t)\,dt$.



Graph of f

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.

(d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

Solution. We see that $g'(x) = f(x)$ for $x \in [-4, 12]$. In particular, g is differentiable on that interval.

(a) We know that g has a relative extremum at x if and only if g' changes sign at x. Since g' does not change sign at x, g has not relative extremum there.

(b) We know that g has a point of inflection at x if and only if g'' changes sign at x. Since g' changes from increasing to decreasing at $x = 4$, g'' changes from positive to negative at $x = 4$, so g changes from concave up to concave down at $x = 4$. So yes, g has a point of inflection there.

(c) The College Board *really* wants you to use EVT here. So say the following.

We know that g has extreme values at an endpoint, or at a critical point where the derivative changes sign. The endpoints are $x = -4$ and $x = 12$, the pertinent critical points at $x = -2$ and $x = 6$.

We know that $g(2) = \int_2^2 f(x)\,dx$, so $g(2) = 0$. We compute the values at the other points by adding the area under the curve as we move rightward, or subtracting if we move leftward. These are all triangles, so its not terribly difficult to see that

- $g(-2) = -8$
- $g(-6) = -4$
- $g(6) = 8$
- $g(10) = 0$
- $g(12) = -4$.

Thus the minimum value is -8 at $x = -2$, and the maximum value is 8 at $x = 6$.

(d) We see that $g(x) = 0$ if $x = 2$ or $x = 10$, so $g(x) \leq 0$ if $x \in [-4, 2] \cup [10, 12]$.

□

Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$

$f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$

$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$

(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$.
Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.

2 : answer with justification

(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint.

$f'(x) = 0 \Rightarrow x = -2, x = 2$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

(d) $f''(-5) = \dfrac{2 - 0}{-6 - (-2)} = -\dfrac{1}{2}$

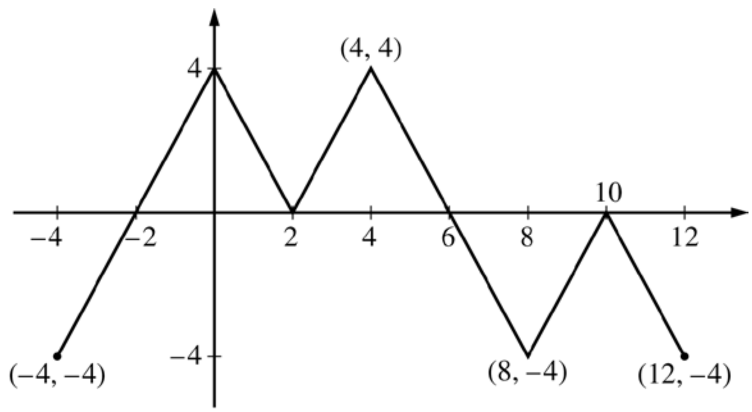
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3} = -1$

$f''(3)$ does not exist because
$\lim\limits_{x \to 3^-} \dfrac{f'(x) - f'(3)}{x - 3} \neq \lim\limits_{x \to 3^+} \dfrac{f'(x) - f'(3)}{x - 3}$.

$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad\ \text{with explanation} \end{cases}$

Question 3

The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.



Graph of f

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

---

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
−4	−4
−2	−8
6	8
12	−4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

$$4 : \begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \text{ as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$$

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals