Vector Calculus — Responses 0421

Dr. Paul L. Bailey — Monday, April 21, 2020

Question 1. I'm confused. Are all sentences false? A question is a sentence and a question can't be "false", right?

Proof. The sentence "All sentences are false" is not a paradox. It is a false sentence.

Here is a true sentence: "A horse is a horse." Thus, it is not the case the all sentences are false. □

Question 2. How do you do problem 16?

Problem 1 (Thomas §16.5 # 16)**.** Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, $z = \pm c$.

Solution. We use the formula

$$\texttt{Surface Integral} = \iint_S g\, d\sigma = \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA.$$

We do this one face at a time.

Let us start with the face $z = c$. Call it S. Then S is part of the level surface of the function $f(x, y, z) = z$ where $f(x, y, z) = c$. Then $\nabla f = \langle 0, 0, 1\rangle$, so $|\nabla f| = 1$.

A plane perpendicular to this is the xy-plane, so let R be the projection of S onto the xy-plane. Then $R = [-a, b] \times [-b, b]$ is a rectangle in the xy-plane. In this case, $\vec{p} = \langle 0, 0, 1\rangle$, so $|\nabla f \cdot \vec{p}| = |1| = 1$.

Now, with $g(x, y, z) = xyz$, we have

$$\begin{aligned}
\texttt{Surface Integral} &= \iint_S g\, d\sigma \\
&= \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA \\
&= \iint_R xyz \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\, dA \\
&= \iint_R xyz \frac{1}{1}\, dA \\
&= \int_{-a}^{a} \int_{-b}^{b} xyc\, dy\, dx \\
&= 0
\end{aligned}$$

By symmetry, we see that the value on each of the sides of the rectangular box will be zero, so the total integral is zero. □

Question 3. Can you go over the solution to number 26?

Problem 2 (Thomas §16.5 # 26)**.** Find the flux of the field $\vec{F}$ across the portion of the sphere $x^2+y^2+z^2 = a^2$ in the first octant in the direction away from the origin, where

$$\vec{F} = \frac{1}{\sqrt{x^2+y^2+z^2}}\langle x, y, z\rangle.$$

Solution. We know

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,d\sigma,$$

where

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}dA,$$

and f is a function of x, y, and z such that S is a level surface of f, and $\vec{p}$ is a unit normal vector to a plane onto which S projects injectively.

In this case, $f(x,y,z) = x^2+y^2+z^2$, so that S is the locus of $f(x,y,z) = a^2$. Then $\nabla f = \langle 2x, 2y, 2z\rangle$. We take the plane to be the xy-plane, so $\vec{p} = \langle 0,0,1\rangle$. Thus $\nabla f\cdot\vec{p} = 2z$, which is nonnegative in the first octant.

We need that normal vector, which is

$$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}.$$

Thus $\vec{F} = \vec{n}$, and

$$\vec{F}\cdot\vec{n} = |\vec{n}|^2 = 1^2 = 1.$$

Oh, goody.

Let's compute $d\sigma$:

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\,dA = \frac{\sqrt{x^2+y^2+z^2}}{2z}\,dA.$$

Thus,

$$\begin{aligned}
\text{Flux} &= \iint_S \vec{F}\cdot\vec{n}\,d\sigma \\
&= \iint_S 1\cdot\frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\,dA \\
&= \iint_S \frac{\sqrt{x^2+y^2+z^2}}{2z}\,dA \\
&= \iint_S \frac{a}{2\sqrt{a^2-x^2-y^2}}\,dA \quad \text{(along the surface } S\text{)} \\
&= \frac{1}{2}\int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2-r^2}}r\,dr\,d\theta \quad \text{(converting to polar)} \\
&= \frac{1}{2}\int_0^{\pi/2} d\theta\cdot\left(-\frac{a}{2}\right)\int_0^a \frac{-2\,dr}{\sqrt{a^2-r^2}} \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{2}{1}\right)\sqrt{a^2-r^2}\Big|_0^a \\
&= \frac{a^2\pi}{2}.
\end{aligned}$$

□