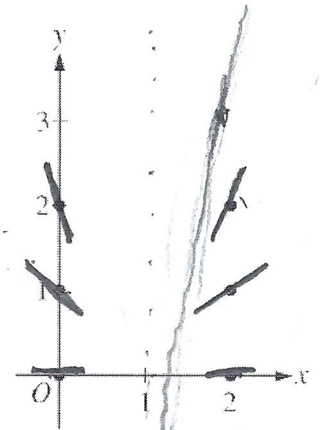
AP CALCULUS AB
DR. PAUL L. BAILEY

Homework 0423d
Thursday, April 23, 2020

Problem 1. Consider the differential equation $\boxed{\frac{dy}{dx} = \frac{y^2}{x-1}}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



x	y	dy/dx
0	0	0
0	1	−1
0	2	−4
2	0	0
2	1	1
2	2	4

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

Point $(2,3)$ $\quad y = m(x-2)+3$

$$\frac{dy}{dx} = \frac{3^2}{2-1} = 9 \qquad \text{Line is}$$

$$L(x) = \boxed{y = 9(x-2)+3}$$

$$f(2.1) \approx L(2.1) = 9(0.1)+3 = \boxed{3.9}$$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$\int \frac{dy}{y^2} = \int \frac{dx}{x-1}$$

$$\boxed{-\frac{1}{y} = \ln(x-1) + C}$$

so $-\frac{1}{3} = \ln(2-1) + C$

so $C = -\frac{1}{3}$

$-\frac{1}{y} = \ln(x-1) - \frac{1}{3}$, so $\boxed{y = \frac{1}{\frac{1}{3} - \ln(x-1)}}$

Problem 2. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

$W(0) = 1400$

$\frac{dW}{dt}(0) = \frac{1}{25}(1400 - 300) = \frac{1100}{25} = 44$

$L(t) = m(t - t_0) + W_0$

$= 44t + 1400$

$L(\frac{1}{4}) = 11 + 1400$

$= \boxed{1411}$

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

$\frac{d^2w}{dt^2} = \frac{d}{dt}\frac{dW}{dt}$

$= \frac{d}{dt}\frac{1}{25}(w - 300)$

$= \frac{1}{25}\frac{dw}{dt} = \frac{1}{25}\cdot\frac{1}{25}(W - 300) = \frac{1}{625}(w - 300)$

we know W is concave up if $\frac{d^2w}{dt^2} > 0$. This is true since w is increasing.

We So $L(\frac{1}{4}) \leq W(\frac{1}{4})$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

$\int \frac{dw}{w - 300} = \int \frac{1}{25} dt$

$\ln(w - 300) = \frac{t}{25} + C$

at $(0, 1400)$: $\ln(1100) = C$

So $\ln(w - 300) = \frac{t}{25} + \ln(1100)$

Thus

$W - 300 = e^{\frac{t}{25} + \ln(1100)} = e^{t/25} e^{\ln(1100)}$

So $\boxed{W = 300 + 1100 e^{t/25}}$

Question 4

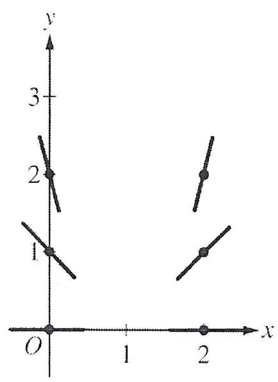
Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} = \frac{3^2}{2-1} = 9$

An equation for the tangent line is $y = 9(x-2) + 3$.

$f(2.1) \approx 9(2.1 - 2) + 3 = 3.9$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\frac{1}{y^2}\,dy = \frac{1}{x-1}\,dx$

$\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$

$-\frac{1}{y} = \ln|x-1| + C$

$-\frac{1}{3} = \ln|2-1| + C \Rightarrow C = -\frac{1}{3}$

$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$

$y = \dfrac{1}{\frac{1}{3} - \ln(x-1)}$

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2: \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \ge 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \le t \le \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2: \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \le t \le 20$

$5: \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

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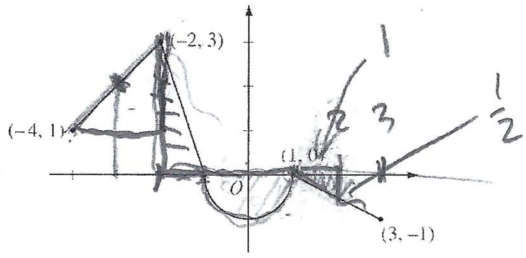
Problem 1. Let f be a continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given below.



Graph of f

$A_\Delta = \frac{1}{2}(1)(\frac{1}{2})$

Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

We know $g' = f$ and $g(1) = 0$.

$g(2) = g(1) + \int_1^2 f(t)\,dt$

$= 0 - \frac{1}{4} = -\frac{1}{4}$

$g(-2) = 0 + \underset{[-1,1]}{\text{area semicircle}} - \underset{[-2,-1]}{\text{area } \Delta}$

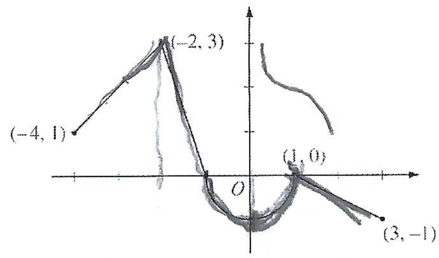
$= \boxed{\frac{\pi}{2} - \frac{3}{2}}$

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

$g'(-3) = 2.$

$g''(-3) = 1.$

Problem 1 (continued). Let f be a continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given below.



Graph of f

Let g be the function given by $g(x) = \int_1^x f(t)\, dt$.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

We know g has a horizontal tangent when $g' = 0$. We have a graph of g'.

Look where it is zero.
This occurs at $x = -1$ and $x = 1$
Since g' does not change sign at $x = 1$, g does not have a local extremum there.
Since g' changes from positive to negative at $x = -1$, g changes from increasing to decreasing at $x = -1$,

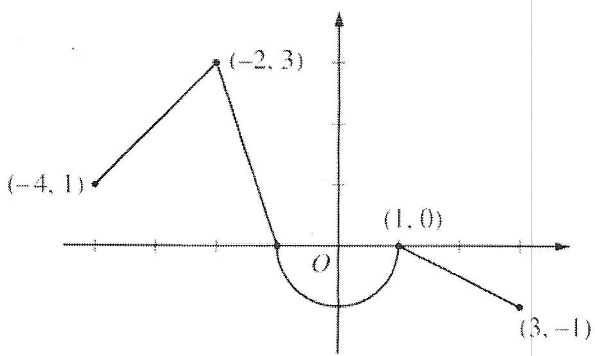
(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

so g has a local maximum at $x = -1$.

We know g has a poi at x if the concavity of g changes at x, so g'' changes sign at x, so g' changes direction, from increasing to decreasing or decreasing to increasing.
This occurs at $x = -2, 0, 1$.

Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.



Graph of f

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$

$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

0 points for BALD ANSWER

(b) $\cancel{g'(x) = f(x) \Rightarrow g'(-3) = f(-3)} = 2$

$g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$