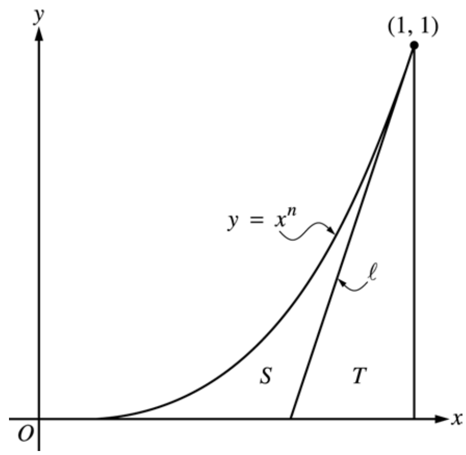
AP Calculus AB — Homework 0427 - Solutions — Name:

Dr. Paul L. Bailey — Monday, April 27, 2020

**Problem 1.** Let $\ell$ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown below.




**(a)** Find $\int_0^1 x^n \, dx$ in terms of $n$.

*Solution.* By the power rule,

$$\int_0^1 x^n \, dx = \left. \frac{x^{n+1}}{n+1} \right|_0^1 = \boxed{\frac{1}{n+1}}.$$

□

**(b)** Let $T$ be the triangular region bounded by $\ell$, the $x$-axis, and the line $x = 1$. Show that the area of $T$ is $\frac{1}{2n}$.

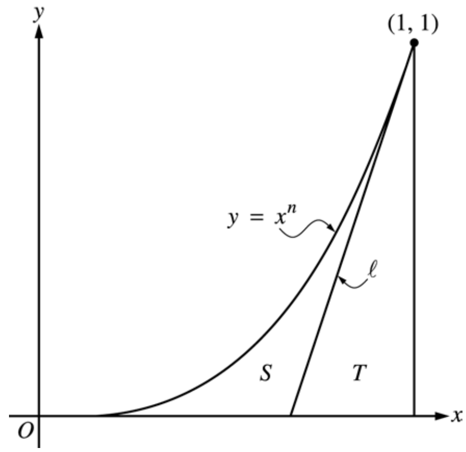
*Solution.* The area of the triangle is $\text{area}(T) = \frac{1}{2}bh$, where the height is $h = 1$ and the base $b$ needs to be determined. To do this, we find the equation of the line, then its $x$-intercept.

The slope of the line is $\left. \frac{d}{dx} x^n \right|_{x=1} = \left. nx^{n-1} \right|_{x=1} = n$. Thus the line is $y = n(x-1)+1$. Setting $y = 0$ and solving for $x$ yields $x = 1 - \frac{1}{n}$. Thus $b = 1 - (1 - \frac{1}{n}) = \frac{1}{n}$, whence

$$\boxed{\text{area}(T) = \frac{1}{2n}.}$$

□

**Problem 1** (continued). Let $\ell$ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown below.




**(c)** Let $S$ be the region bounded by the graph of $y = x^n$, the line $\ell$, and the $x$-axis. Express the area of $S$ in terms of $n$ and determine the value of $n$ that maximizes the area of $S$.

*Solution.* We have

$$\text{area}(S) = \int_0^1 x^n \, dx - \text{area}(T) = \frac{1}{n+1} - \frac{1}{2n}.$$

No where in the problem does it say $n$ is an integer, so we assume $n$ is a continuous variable; that is, in order to maximize area$(S)$, we take it's derivative with respect to $n$, set it to zero, and solve:

$$\frac{d}{dn}\,\text{area}(S) = \frac{d}{dn}\left(\frac{1}{n+1} - \frac{1}{2n}\right) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0 \quad \Rightarrow \quad n^2 - 2n - 1 = 0.$$

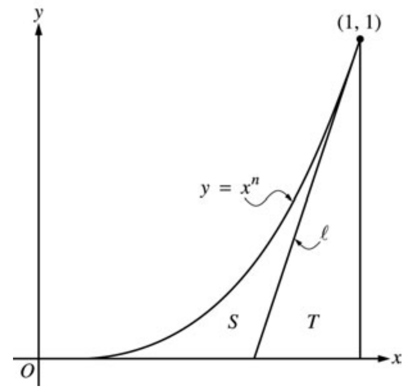
Use the quadratic formula to arrive at

$$\boxed{n = 1 + \sqrt{2}.}$$

□

## Question 6

Let $\ell$ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.




(a) Find $\int_0^1 x^n\,dx$ in terms of $n$.

(b) Let $T$ be the triangular region bounded by $\ell$, the $x$-axis, and the line $x = 1$. Show that the area of $T$ is $\frac{1}{2n}$.

(c) Let $S$ be the region bounded by the graph of $y = x^n$, the line $\ell$, and the $x$-axis. Express the area of $S$ in terms of $n$ and determine the value of $n$ that maximizes the area of $S$.

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(a) $\displaystyle\int_0^1 x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$

$2 : \begin{cases} 1 : \text{antiderivative of } x^n \\ 1 : \text{answer} \end{cases}$

(b) Let $b$ be the length of the base of triangle $T$.

$\frac{1}{b}$ is the slope of line $\ell$, which is $n$

$\text{Area}(T) = \frac{1}{2}b(1) = \frac{1}{2n}$

$3 : \begin{cases} 1 : \text{slope of line } \ell \text{ is } n \\ 1 : \text{base of } T \text{ is } \frac{1}{n} \\ 1 : \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $\text{Area}(S) = \displaystyle\int_0^1 x^n\,dx - \text{Area}(T)$

$= \dfrac{1}{n+1} - \dfrac{1}{2n}$

$\dfrac{d}{dn}\text{Area}(S) = -\dfrac{1}{(n+1)^2} + \dfrac{1}{2n^2} = 0$

$2n^2 = (n+1)^2$

$\sqrt{2}\,n = (n+1)$

$n = \dfrac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$

$4 : \begin{cases} 1 : \text{area of } S \text{ in terms of } n \\ 1 : \text{derivative} \\ 1 : \text{sets derivative equal to 0} \\ 1 : \text{solves for } n \end{cases}$