Category Theory — Lesson 0428
Dr. Paul L. Bailey — Tuesday, April 28, 2020

Let E/F be field extension.

- E/F is *finite* if $[E : F]$ is finite.
- E/F is *primitive* if $E = F[\alpha]$ for some α, in which case $[E : F]$ is the degree of the minimum polynomial of α.
- E/F is *algebraic* if every element in E is algebraic over F; that is, for every $\alpha \in E$ there exists $f \in F[X]$ such that $f(\alpha) = 0$. Every finite extension is algebraic.
- E/F is *normal* if every polynomial over F which has a root in E splits in E. It can be shown that E/F is finite and normal if and only if E is a splitting field for a polynomial over F.
- E/F is *separable* if no irreducible polynomial over F has multiple roots in E. It can be shown that finite separable extensions are primitive.
- E/F is *Galois* if it is normal and separable.

Let $\text{Aut}(E)$ denote the set of all automorphisms of E. Let $\text{Aut}(E/F)$ denote the subgroup of $\text{Aut}(E)$ consisting of those automorphisms we fix F pointwise:

$$\text{Aut}(E/F) = \{\phi \in \text{Aut}(E) \mid \phi(x) = x \text{ for all } x \in F\}.$$

If E/F is a normal extension, we may write $\text{Gal}(E/F) = \text{Aut}(E/F)$. In this case, we call $\text{Gal}(E/F)$ the *Galois group* of E/F.

Yesterday's problems were the following.

Problem 1. Let E/F be a finite separable extension.

(a) Show that $|\text{Aut}(E/F)| \leq [E : F]$.

(b) Show that if E/F is Galois, then $|\text{Aut}(E/F)| = [E : F]$.

Today's problems are the following.

Problem 2 (Bilbo's Lemma)**.** Let E/F be a field extension. Let K be a subfield of E which contains F. Let $\alpha \in E$ be algebraic over F. Let $f \in F[X]$ be the minimum polynomial of α over F, and let $g \in K[X]$ be the minimum polynomial of α over K. Show that g divides f in $K[X]$.

Let $H \leq \text{Aut}(E)$. The *fixed field* of H is

$$\text{Fix}(H) = \{x \in E \mid \phi(x) = x \text{ for all } \phi \in H\}.$$

Problem 3. Let E/F be a finite separable extension. Let $H \leq \text{Aut}(E/F)$. Let $K = \text{Fix}(H)$. Show that K is a subfield of E which contains F.

Next is tomorrow's problem, but if you see through it today, we can do more tomorrow! (Hint: Bilbo's Lemma is a lemma for this problem.)

Problem 4. Let E/F be a finite separable extension. Let $H \leq \text{Aut}(E/F)$. Let $K = \text{Fix}(H)$.

(a) Show that if E/F is normal, then E/K is normal.

(b) Show that if E/F is normal, then $\text{Aut}(E/K) = H$.