AP Calculus AB
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Homework 0430d
Wednesday, April 30, 2020

Problem 1. Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table below, where DNE indicates that the derivatives of f do not exist at $x = 2$.

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	-1	Negative	0	Positive	2	Postive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

We know f has a critical point at x if
f'(x) = 0 or f'(x) DNE.
We have critical points at x = 1 and x = 2.
At x = 1, f' does not change sign, so f does not have a local extremum there.
At x = 2, f' changes sign from positive to negative,
so f changes from increasing to decreasing,
So we have a local maximum at x = 2.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of f.

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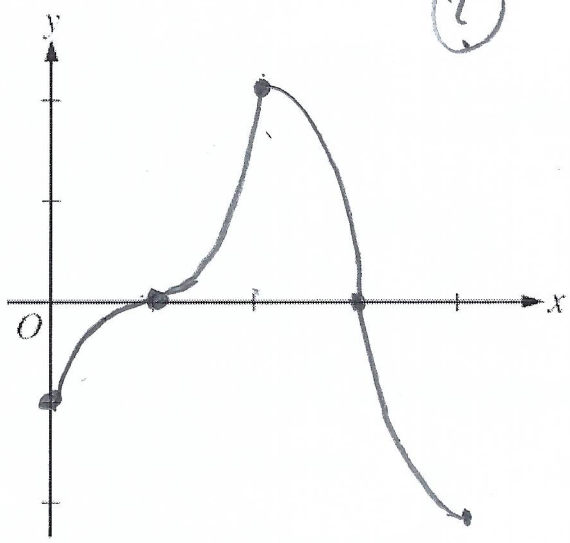


Problem 1 (continued). Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table below, where DNE indicates that the derivatives of f do not exist at $x = 2$.

g'

g''

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	-1	Negative	0	Positive	2	Postive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive

(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.

We see that $g' = f$.

We seek $g'(x) = 0$ ...

This occurs at $x = 1$ and $x = 3$.

At $x = 1$, g' changes from negative to positive,

so g has a local minimum at $x = 1$.

At $x = 3$, g' changes from positive to negative,

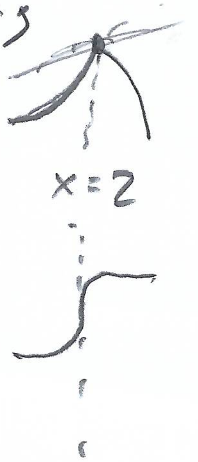
so g has a local maximum there.

(d) For the function g defined in part **(c)**, find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.

We know g has a point of inflection at x if $g'' = f'$ changes sign at x,

~~It is required that~~

and has a well-defined tangent line.



$x = 2$

Does g has a tangent at $x = 2$?

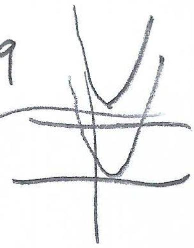
g'' changes sign at $x = 2$, ... we see $g'(2) = 2$.

So yes g has a p.o.i. at $x = 2$.

$t^2 - 2t + 10$

$= t^2 - 2t + 1 + 9$

$= (t-1)^2 + 9$



Problem 1. Two particles move along the x-axis. For $0 \leq t \leq 8$, the position of particle P at time t is given by $x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time t is given by $\boxed{v_Q(t) = t^2 - 8t + 15.}$ Particle Q is at position $x = 5$ at time $t = 0$.

We know P is moving left when the derivative of its position is negative.

(a) For $0 \leq t \leq 8$, when is particle P moving to the left?

$$\frac{d}{dt} x_P = \frac{2t-2}{t^2-2t+10} \cancel{< 0 \text{ when } 2t}$$

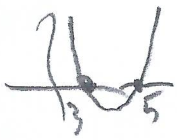
Since $t^2 - 2t + 10 > 0$ all $t \in \mathbb{R}$,

we know $\frac{d}{dt} x_P < 0$ exactly when

$2t - 2 < 0$, that is, $[0, 1)$.

(b) For $0 \leq t \leq 8$, find all times t during which the two particles travel in the same direction.

This occurs when v_P and v_Q have the same sign.



Now $v_Q(t) = t^2 - 8t + 15$

$= (t-3)(t-5)$.

Let's draw sign charts

	0		1		3		5		8
v_P		$-$		$+$		$+$		$+$	
v_Q		$+$		$+$		$-$		$+$	

We see the signs are the same on

$(1,3) \cup (5, 8)$.

Problem 1. Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle P at time t is given by $x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time t is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(c) Find the acceleration of particle Q at time $t = 2$. Is the speed of particle Q increasing, decreasing, or neither at time $t = 2$? Explain your reasoning.

$$a_Q(t) = \frac{dv_Q}{dt} = 2t - 8.$$

$$a_Q(2) = 4 - 8 = -4 < 0$$

but $v_Q(2) = 4 - 16 + 15 = 3 > 0$

We know that speed is increasing when the sign of velocity and acceleration are the same, and decr. if they are different. Here they are different, so, decreasing.

(d) Find the position of particle Q the first time it changes direction.

The first time V_Q changes sign is $t=3$.

$$x_Q(3) = x_Q(0) + \int_0^3 v_Q(t)\,dt \quad \text{by } \underline{\text{FTC}}$$

$$= 5 + \int_0^3 t^2 - 8t + 15\,dt$$

$$= 5 + \frac{t^3}{3} - \frac{8t^2}{2} + 15t \Big|_0^3$$

$$= 5 + 9 - 36 + 45$$

$$= 23$$

Question 4

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
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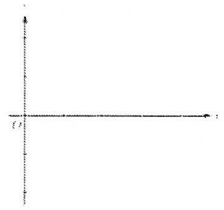
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive

Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exist at $x = 2$.

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

(b) On the axes provided, sketch the graph of a function that has all the characteristics of f.
(Note: Use the axes provided in the pink test booklet.)



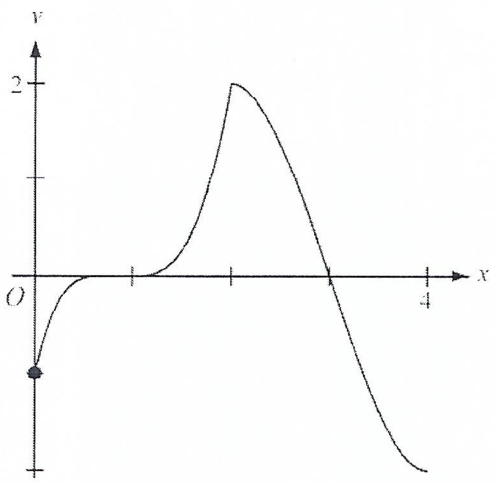
(c) Let g be the function defined by $g(x) = \int_1^x f(t)\,dt$ on the open interval $(0, 4)$. For $0 < x < 4$, find all values of x at which g has a relative extremum. Determine whether g has a relative maximum or a relative minimum at each of these values. Justify your answer.

(d) For the function g defined in part (c), find all values of x, for $0 < x < 4$, at which the graph of g has a point of inflection. Justify your answer.

(a) f has a relative maximum at $x = 2$ because f' changes from positive to negative at $x = 2$.

$2 : \begin{cases} 1 : \text{relative extremum at } x = 2 \\ 1 : \text{relative maximum with justification} \end{cases}$

(b)



$2 : \begin{cases} 1 : \text{points at } x = 0, 1, 2, 3 \\ \quad \text{and behavior at } (2, 2) \\ 1 : \text{appropriate increasing/decreasing} \\ \quad \text{and concavity behavior} \end{cases}$

(c) $g'(x) = f(x) = 0$ at $x = 1, 3$.
g' changes from negative to positive at $x = 1$ so g has a relative minimum at $x = 1$. g' changes from positive to negative at $x = 3$ so g has a relative maximum at $x = 3$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{critical points} \\ 1 : \text{answer with justification} \end{cases}$

(d) The graph of g has a point of inflection at $x = 2$ because $g'' = f'$ changes sign at $x = 2$.

$2 : \begin{cases} 1 : x = 2 \\ 1 : \text{answer with justification} \end{cases}$

Question 5

(a) $x_P'(t) = \dfrac{2t-2}{t^2-2t+10} = \dfrac{2(t-1)}{t^2-2t+10}$

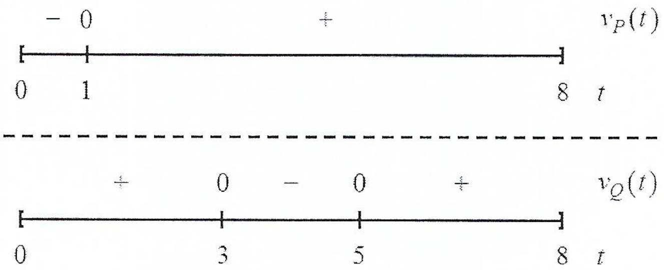
$t^2 - 2t + 10 > 0$ for all t.

$x_P'(t) = 0 \Rightarrow t = 1$
$x_P'(t) < 0$ for $0 \le t < 1$.

Therefore, the particle is moving to the left for $0 \le t < 1$.

$2: \begin{cases} 1: x_P'(t) \\ 1: \text{interval} \end{cases}$

(b) $v_Q(t) = (t-5)(t-3)$
$v_Q(t) = 0 \Rightarrow t = 3, t = 5$



Both particles move in the same direction for $1 < t < 3$ and $5 < t \le 8$ since $v_P(t) = x_P'(t)$ and $v_Q(t)$ have the same sign on these intervals.

$2: \begin{cases} 1: \text{intervals} \\ 1: \text{analysis using } v_P(t) \text{ and } v_Q(t) \end{cases}$

Note: $1/2$ if only one interval with analysis

Note: $0/2$ if no analysis

(c) $a_Q(t) = v_Q'(t) = 2t - 8$
$a_Q(2) = 2 \cdot 2 - 8 = -4$

$a_Q(2) < 0$ and $v_Q(2) = 3 > 0$

At time $t = 2$, the speed of the particle is decreasing because velocity and acceleration have opposite signs.

$2: \begin{cases} 1: a_Q(2) \\ 1: \text{speed decreasing with reason} \end{cases}$

(d) Particle Q first changes direction at time $t = 3$.

$$x_Q(3) = x_Q(0) + \int_0^3 v_Q(t)\,dt = 5 + \int_0^3 \left(t^2 - 8t + 15\right)dt$$
$$= 5 + \left[\frac{1}{3}t^3 - 4t^2 + 15t\right]_{t=0}^{t=3} = 5 + (9 - 36 + 45) = 23$$

$3: \begin{cases} 1: \text{antiderivative} \\ 1: \text{uses initial condition} \\ 1: \text{answer} \end{cases}$