AP Calculus AB
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Homework 0504
Monday, May 4, 2020

Problem 1. Let $f : [0,5] \to \mathbb{R}$ be given by $f(x) = x^3 - 3x + 1$.

(a) Sketch the graph of f as follows. Let $g(x) = \boxed{x^3 - 3x + 2}$. Factor it. Sketch it by first plotting its zeros. Now $f(x) = g(x) - 1$. Shift the graph of g down by one.

$g(1) = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

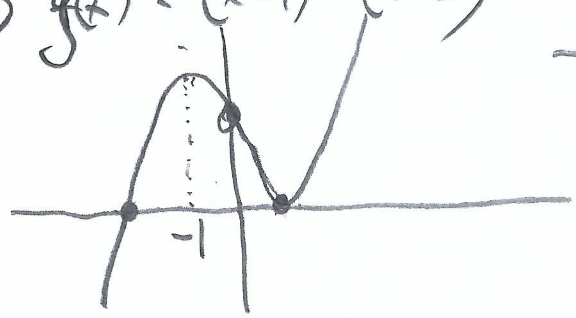
$x^2 + x - 2 = (x-1)(x+2)$

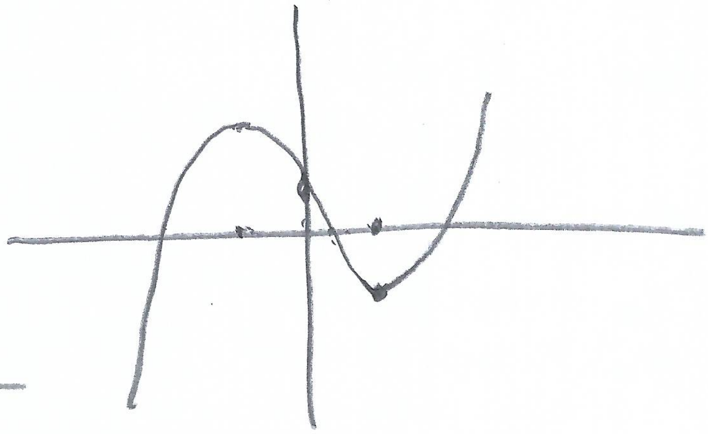
So $g(x) = (x-1)^2(x+2)$

$g'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1$

$x = \pm 1$

So $f(x) = g(x) - 1$





(b) What is the hypothesis of the Intermediate Value Theorem (IVT)? Does f satisfy it? Show that there exists $c \in (0,5)$ such that $f(c) = 0$.

We know $f(x) = x^3 - 3x + 1$ is continuous on $[0,5]$.

Note $f(0) = 1$ and $f(1) = 1 - 3 + 1 = -1$.

Since f is continuous from $x=0$ to $x=1$,

we know, by IVT, $f(c) = 0$ for some $c \in (0,1)$.

Problem 1 (continued). Let $f : [0, 5] \to \mathbb{R}$ be given by $f(x) = x^3 - 3x + 1$.

(c) What is the hypothesis of the Extreme Value Theorem (EVT)? Does f satisfy it? Find $c_1, c_2 \in [0, 5]$ such that f has an absolute minimum at c_1 and an absolute maximum at c_2.

Note f is continuous on $[0,5]$, and so has absolute extrema.

Note $f'(x) = 3x^2 - 3$, so if $f'(x) = 0$, $x = \pm 1$.

But we are only considering the interval $[0,5]$, so the only c.p. there is $x = 1$.

Now $f(0) = 1$

$f(5) = 125 - 15 + 1 = 111.$

$f(1) = -1.$

So f has an absolute min val of -1 at $x = 1$ and f has an abs max val of 111 at $x = 5$.

(d) What is the hypothesis of the Mean Value Theorem (MVT)? Does f satisfy it? What is the conclusion of MVT? Find $c \in (0, 5)$ such that f satisfies the conclusion of the MVT at $x = c$.

We want $c \in (0,5)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Now $\frac{f(5) - f(0)}{5 - 0} = \frac{111}{5}$

and $f'(c) = 3c^2 - 3$

So $3c^2 - 3 = \frac{111}{5}$, so $c^2 - 1 = \frac{37}{5}$

So $c^2 = \frac{42}{5}$

So $\boxed{c = \sqrt{\frac{42}{5}}}$

Problem 2. Let $f : [0,5] \to \mathbb{R}$ be the piecewise defined function given by

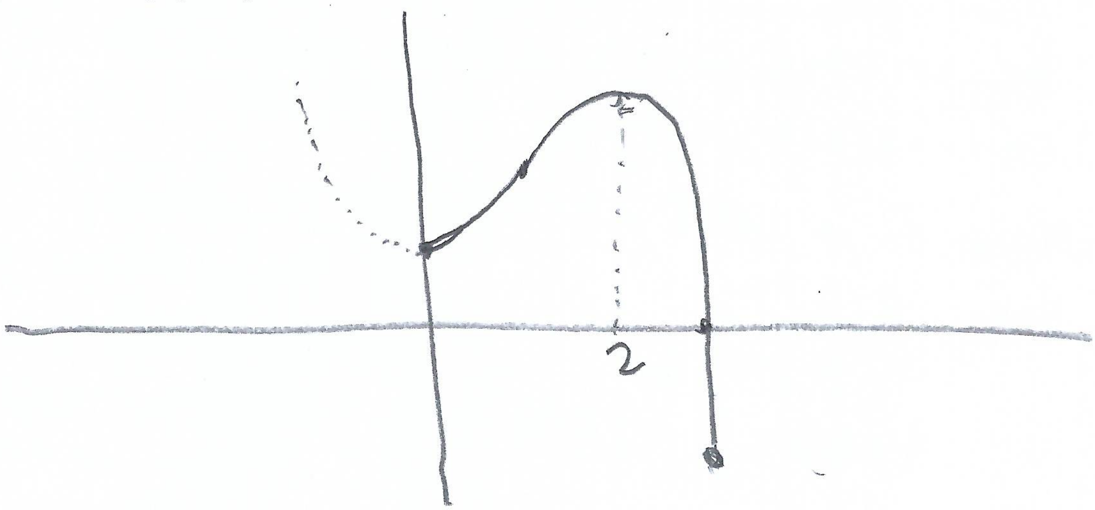
$$f(x) = \begin{cases} x^2+1 & \text{if } x \in [0,1)\,; \\ 3-(x-2)^2 & \text{if } x \in [1,5]\,. \end{cases}$$

$f(5) = 3-(5-2)^2$
$= -6$

For $x \in (0,1) \cup (1,5)$, the derivative of f is

$$f'(x) = \begin{cases} 2x & \text{if } x \in [0,1)\,; \\ -2(x-2) & \text{if } x \in [1,5]\,. \end{cases}$$

(a) Sketch the graph of f.



(b) Does f satisfy the hypothesis of IVT on $[0,5]$? Show that there exists $c \in (0,5)$ such that $f(c) = 0$.

Note f is continuous at $x=1$, since

$$x^2+1\Big|_{x=1} = 2 = 3-(x-2)^2\Big|_{x=1}$$

We know f is continuous at $x=1$

since $\lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$.

Since f is continuous on $[0,5]$ and

$f(0) = 1$ and $f(5) = -6$,

there exists $c \in (0,5)$ such that $f(c)=0$,

by IVT.

Problem 2 (continued). Let $f : [0,5] \to \mathbb{R}$ be the piecewise defined function given by

$$f(x) = \begin{cases} x^2+1 & \text{if } x \in [0,1)\ ; \\ 3-(x-2)^2 & \text{if } x \in [1,5]\ . \end{cases}$$

For $x \in (0,1) \cup (1,5)$, the derivative of f is

$$f'(x) = \begin{cases} 2x & \text{if } x \in [0,1)\ ; \\ -2(x-2) & \text{if } x \in [1,5]\ . \end{cases}$$

(c) Does f satisfy the hypothesis of EVT on $[0,5]$? Find $c_1, c_2 \in [0,5]$ such that f has an absolute minimum at c_1 and an absolute maximum at c_2.

The critical points occur when $f'(x)=0$,
so $-2(x-2)=0$, so $x-2=0$, so $x=2$.
Now f is diff at $x=1$, since $2(1) = -2(1-2)$.
Now $f(0)=1$
$f(5)=-6$
$f(2) = 3-(2-2)^2 = 3$

Thus f has an abs min at $x=5$ and an abs max at $x=2$.

(d) Does f satisfy the hypothesis of MVT on $[0,5]$? If so, find $c \in (0,5)$ such that f satisfies the conclusion of the MVT at $x=c$.

f is cont on $[0,5]$ and diff on $(0,5)$,
so MVT applies.

Now $\dfrac{f(5)-f(0)}{5-0} = \dfrac{-6-1}{5} = -\dfrac{7}{5}$

since $x^2+1>0$, the c we seek is on the right side of $x=1$.
So, $f'(c) = -2(c-2) = -\dfrac{7}{5}$

so $c-2 = \dfrac{7}{10}$

so $\boxed{c = \dfrac{27}{10}}$