AP Calculus AB
Dr. Paul L. Bailey

Homework 0507e
Thursday, May 7, 2020

Problem 1. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table below.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

We have $H'(6) \approx \dfrac{H(7)-H(5)}{7-5} = \dfrac{11-6}{2} = \dfrac{5}{2}$

Here, $H'(6)$ is the instantaneous rate of change of the height of the tree in $\dfrac{\text{meters}}{\text{year}}$.

(b) Explain why there must be at least one time t, $2 < t < 10$, such that $h'(t) = 2$.

since H is twice-differentiable, we know H is continuous on $[3,5]$ and differentiable on $(3,5)$, so there exists $c \in (3,5)$ such that $H'(c) = \dfrac{H(5)-H(3)}{5-3} = \dfrac{6-2}{5-3} = 2\,\frac{m}{yr}$

7.5

Problem 1 (continued). The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table below.

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

Trapezoidal: $\frac{1.5+2}{2}(3-2) + \frac{2+6}{2}(5-3) + \frac{6+11}{2}(7-5) + \frac{11+15}{2}(10-7) = 65.75$

Left: $1.5(3-2) + 2(5-3) + 6(7-5) + 11(10-7)$

Right: $2(3-2) + 6(5-3) + 11(7-5) + 15(10-7)$

Midpoint: $H(3)(5-2) + H(7.5)(10-5)$

Average Height $= \frac{1}{10-2}\int_2^{10} H(t)\,dt \approx \frac{1}{8}(65.75) = \frac{263}{32}$ m

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to the model, what is the rate of change of the height of the tree with respect to time, in meter's per year, at the time when the tree is 50 meters tall?

Know $\frac{dx}{dt} = 0.03 \frac{m}{yr}$. Chase: $\frac{dG}{dt}$ when $G = 50$

$\frac{dG}{dt} = \frac{d}{dt}\frac{100x}{1+x}$

$= \frac{100(1+x) - 100x}{(1+x)^2} \cdot \frac{dx}{dt}$

$= \frac{200-100}{4}(0.03)$

$= \boxed{0.75} \frac{m}{yr}$

$50 = \frac{100x}{1+x}$

$50 + 50x = 100x$

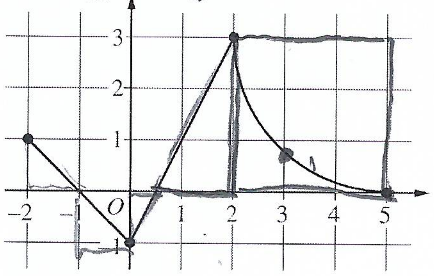
$50 = 50x$

$x = 1$

AP CALCULUS AB
DR. PAUL L. BAILEY

Homework 0507f
Thursday, May 7, 2020

Problem 1. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure below shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.



Graph of f

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$$

$$= 7 - \left[\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{9}{4} + \left[9 - \tfrac{9}{4}\pi\right]\right]$$

$$= 7 - 2 - 9 + \tfrac{9}{4}\pi$$

$$= \boxed{\tfrac{9}{4}\pi - 4}$$

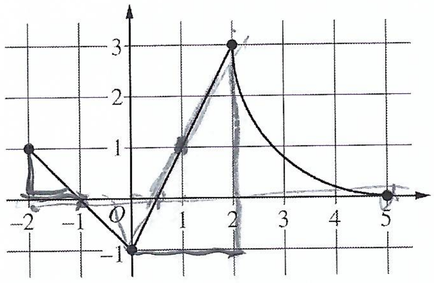
(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$. By FTC,

$$= 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$$

$$= 2[f(5) - f(3)] + 8$$

$$= 2[0 - 3 - \sqrt{5}] + 8 = 2\sqrt{5} + 2$$

Problem 1 (continued). The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure below shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.



Graph of f

what is $f'(1)$?

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

We see that $g' = f$ on $(-2, 5)$
Since g is differentiable, its critical points occur at x if $g'(x) = 0$.
This occurs when $x = -1$ or $x = \frac{1}{2}$.

$g(-2) = 0$
$g(5) = 11 - \frac{9}{4}\pi$
$g(-1) = \frac{1}{2}$
$g(\frac{1}{2}) = -\frac{1}{4}$

we see that the maximum value of g is $11 - \frac{9}{4}\pi$ at $x = 5$.

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$$= \frac{10^1 - 3(2)}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}} = \frac{16}{4 - \pi}$$

Question 4

(a) $H'(6) \approx \dfrac{H(7) - H(5)}{7 - 5} = \dfrac{11 - 6}{2} = \dfrac{5}{2}$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\dfrac{H(5) - H(3)}{5 - 3} = \dfrac{6 - 2}{2} = 2$

Because H is differentiable on $3 \le t \le 5$, H is continuous on $3 \le t \le 5$.

By the Mean Value Theorem, there exists a value c, $3 < c < 5$, such that $H'(c) = 2$.

$2 : \begin{cases} 1 : \dfrac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using Mean Value Theorem} \end{cases}$

(c) The average height of the tree over the time interval $2 \le t \le 10$ is given by $\dfrac{1}{10 - 2}\displaystyle\int_2^{10} H(t)\, dt$.

$$\frac{1}{8}\int_2^{10} H(t)\, dt \approx \frac{1}{8}\left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3\right)$$
$$= \frac{1}{8}(65.75) = \frac{263}{32}$$

The average height of the tree over the time interval $2 \le t \le 10$ is $\dfrac{263}{32}$ meters.

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{approximation} \end{cases}$

(d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1 + x)100 - 100x \cdot 1}{(1 + x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1 + x)^2} \cdot \frac{dx}{dt}$$

$$\left.\frac{d}{dt}(G(x))\right|_{x=1} = \frac{100}{(1 + 1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\dfrac{3}{4}$ meter per year.

$3 : \begin{cases} 2 : \dfrac{d}{dt}(G(x)) \\ 1 : \text{answer} \end{cases}$

Note: max 1/3 [1-0] if no chain rule

Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$3: \begin{cases} 1: \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1: \int_{-2}^{5} f(x)\,dx \\ 1: \text{answer} \end{cases}$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$= 2(f(5) - f(3)) + 4(5 - 3)$

$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$

$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = \left[2f(x) + 4x\right]_{x=3}^{x=5}$

$= (2f(5) + 20) - (2f(3) + 12)$

$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$

$= 2 + 2\sqrt{5}$

$2: \begin{cases} 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$3: \begin{cases} 1: g'(x) = f(x) \\ 1: \text{identifies } x = -1 \text{ as a candidate} \\ 1: \text{answer with justification} \end{cases}$

(d) $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$

$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer